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A TRUNCATED STATISTICAL MODEL FOR ANALYZING THE SIZE-EFFECT ON TENSILE STRENGTH OF CONCRETE STRUCTURES

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Abstract

The statistical model presented in this paper assumes a truncated defect size distribution and predicts a scale effect in complete accordance with that of the Multifractal Scaling Law proposed by Carpinteri (1994a) and Carpinteri et al. (1994a, 1994b, 1995). This truncated distribution model is based on the weakest link concept and aims at evaluating the size effect on the nominal tensile strength of concrete. The expression of the truncated distribution is a Beta-Distribution. The structural failure occurs when the most dangerous defect reaches the propagation critical condition, based on a local LEFM failure criterion. By using this local failure condition, it is possible to obtain the local strength distribution. The global failure probability is obtained by the composition of the n independent local failure conditions. The paper ends with the presentation of the results obtained by this model and with the comparison with some experimental data.

1 Introduction

The classical theories, like Plasticity or Elasticity with an upper bound stress (or strength), do not predict scale effects. The nominal strength, then, can be assumed as a mechanical property of the material, invariant with the structural size.

On the other hand, theories like Linear Elastic Fracture Mechanics and statistical models with unbounded defect size distribution (e.g. self-similarity distribution) predict a linear scale effect law in the bilogarithmic plane $\ln\sigma - \ln d$. The same trend can be obtained using the Damage Theory, Carpinteri (1986), the Renormalization Group Theory, Carpinteri (1994a), and the monofractal hypothesis for the physical dimension of the microstructure, Carpinteri (1994b). The exponent of the power law is the slope of the straight line in the diagram $\ln\sigma - \ln d$. This slope varies between zero, for the theories that do not predict any scale effect (e.g. constant strength), and the upper bound 1/2 for Linear Elastic Fracture Mechanics. The statistical model presented by Carpinteri (1986) for large dispersion of defects (e.g. very disordered material microstructures) predicts a power law with an exponent close to the upper bound 1/2.

The Multifractal Scaling Law proposed by Carpinteri et al. (1994a, 1994b, 1995) is based on a multifractal hypothesis for material microstructure and seems particularly appropriate when the whole scaling range is considered. In this case, the scale effect in the bilogarithmic plane is not linear. For large structural dimensions a non-zero asymptotic strength is achieved, whereas for small structural dimensions the law tends asymptotically to what is predicted by LEFM.

In this paper a statistical model is presented as an extrapolation of the one based on self-similarity, Carpinteri (1986). This model is based on a truncated dimensional distribution for the equivalent defects. This assumption allows a description of the microstructure in terms of disorder and influence of the disorder at different scales. After a brief review of the self-similarity model, the analytical formulation of the truncated model is provided. Then the results of the parametric analysis are shown as well as the comparison between the new model, the Multifractal Scaling Law and the experimental data of direct concrete tension tests performed at the Politecnico di Torino, Carpinteri and Ferro (1994, 1995).

2 The self-similarity distribution

Carpinteri (1986) considered an isotropic linear elastic body with an initial population of embedded defects. The real imperfections were represented by equivalent defects, i.e. polygonal voids. The interaction among the defects was neglected. According to the weakest link concept, the global condition of failure is reached when the most dangerous defect is in a critical condition. The dimensional distri-

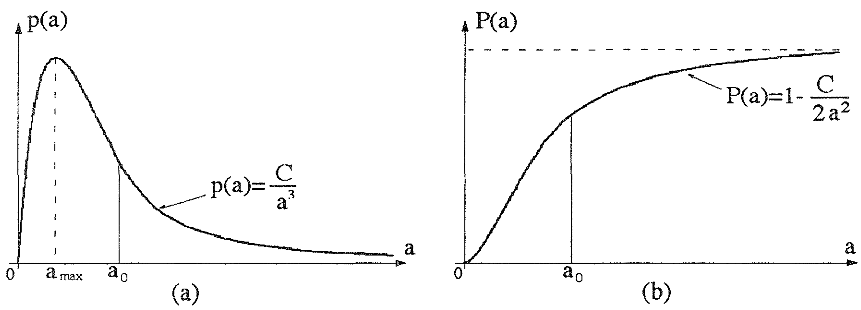
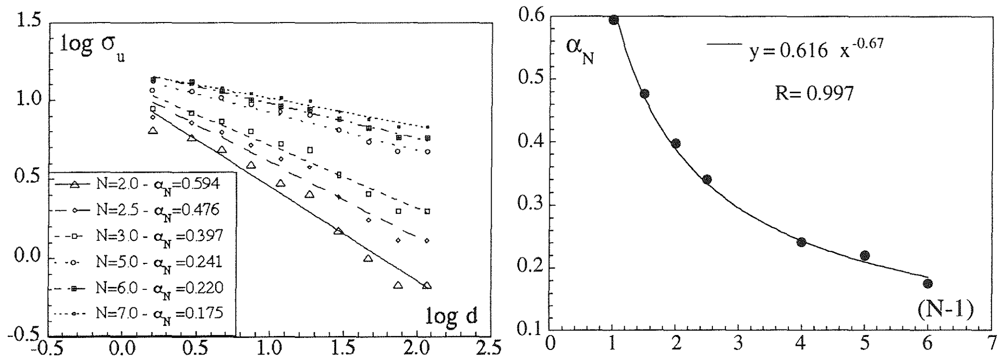


Fig. 1. Defect size distribution of *self-similarity*: (a) probability density; (b) cumulative distribution.



(a) (b)

Fig. 2. (a) Strength size effect laws for a tridimensional body by varying the statistical dispersion parameter N (a_0= 1mm). (b) Power law that relates the slope α_N of the straight line with the parameter N.

bution of self-similarity (Fig.1) is an asymptotic distribution according to which the dimension of the most dangerous defect is proportional to the structural dimension. The self-similarity distribution leads to a strong size effect on the nominal strength, and in the bilogarithmic plane $\ln \sigma - \ln d$, the slope of the straight line is $-1/2$. This slope seems to be a theoretical and experimental upper bound. The self-similarity distribution can be generalized by the introduction of a parameter N. The probability distribution function (PDF) is assumed as follows (Fig. 1a):

$$p(a) = N\frac{C}{a^{N+1}} \quad a > a_0, \tag{1}$$

so that the cumulative distribution function (CDF) becomes (Fig. 1b):

$$P(a) = 1 - \frac{C}{a^N} \quad a > a_0, \tag{2}$$

where a is the defect size, $C = N(1 - P_0)a_0^N$ is a constant, N is the exponent of disorder, a_0 is the defect size beyond which the asymptotic expression is valid.

In this case, the strength size effect can be represented by the $\ln \sigma_u$ versus $\ln d$ straight line with the slope $-\alpha_N$:

$$\alpha_N = \frac{\alpha(\gamma)}{(N-1)^{\varsigma}} \tag{3}$$

where γ is the re-entrant corner angle of the polygonal void while the exponent ζ depends on the secondary features of the material (e.g. density of imperfections, size distribution of the less dangerous defects, etc.).

The self-similarity model has been furthermore developed by Carpinteri and Ferro (1994b) by the introduction of complete expressions for the probability density function and the cumulative density function. The results obtained from this model by a statistical Monte Carlo simulation, varying the degree of disorder, are shown in Fig. 2. In the hypothesis of this distribution, it is possible to affirm that the slope α_N of the size effect law on the tensile strength depends exclusively on the degree of material microstructure disorder, measured by the parameter N. On the other hand, the experimental evidence emerges that the degree of disorder is not a constant property over the entire scaling range. The influence of disorder is strong at the smaller scales and decreases at the larger scales. That corresponds to a size effect that is dominant at the smaller scales and vanishes at the larger scales.

3 The truncated distribution

The goal of the truncated distribution is to take into account the influence of disorder over the entire scaling range. The structural volume is divided in many elementary volumes, each representative of the effective microstructure. The elementary volume is simulated by an equivalent defect population (penny-shaped cracks) embedded in a linear elastic body. The defect population is characterized by a unique random variable that represents the defect size, a, the interaction between the imperfections being neglected and only the imperfections with the most dangerous shape and the most dangerous orientation considered. The distribution of the random variable, i.e. the crack size, can be assumed to be the so called Beta-Distribution. This function is known in the probabilistic field to describe bounded populations and has already been utilized by other authors, Hild and Marquis (1992), to propose an analogous model. The PDF of the Beta-Distribution can be easily written as a function of a/a_M, namely the defect size normalized with respect to the maximum theoretical defect:

$$p\left(\frac{a}{a_M}\right) = \frac{1}{a_M\, B(\alpha+1,\beta+1)}\left(1-\frac{a}{a_M}\right)^{\beta}\left(\frac{a}{a_M}\right)^{\alpha}, \quad 0 \le \frac{a}{a_M} \le 1 \tag{4}$$

The PDF is a polynomial expression with non-zero values comprised between zero and a_M, and it implies the zero probability to find defects larger than a_M. The

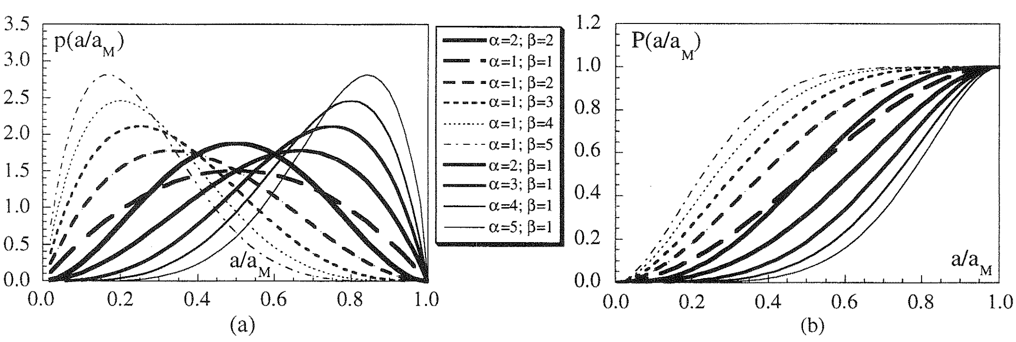


Fig. 3. Truncated defect size distribution by varying the parameters α and β: (a) probability density; (b) cumulative distribution.

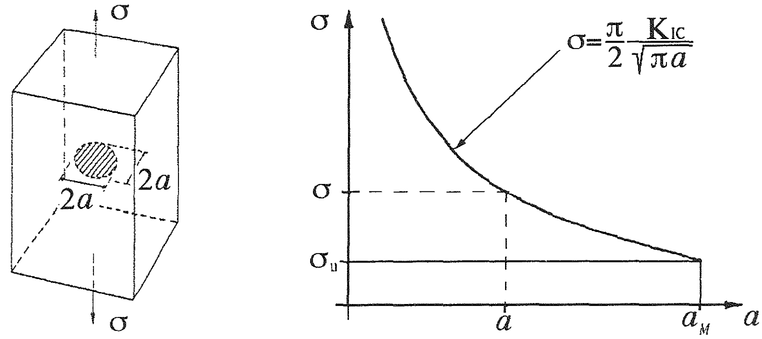


Fig. 4. Local failure criterion (penny-shaped crack) according to LEFM.

parameters α and β determine the shape of the distribution (Fig. 3); from a mathematical point of view, they represent the infinitesimal order of PDF, for the a/a_M ratio tending to zero and to 1, respectively. The CDF can not be found in closed form. It is possible to obtain the following expression only if the values of the parameters α and β are integer:

$$P\left(\frac{a}{a_M}\right) = \frac{1}{B(\alpha+1,\beta+1)} \sum_{i=0}^{\beta}(-1)^i \binom{\beta}{i} \frac{1}{\alpha+i+1} \left(\frac{a}{a_M}\right)^{\alpha+i+1}, \quad 0 \le \frac{a}{a_M} \le 1, \qquad (5)$$

where $\binom{\beta}{i} = \frac{\beta!}{i!(\beta-i)!}$ is the binomial coefficient while $B(\alpha+1,\beta+1)$ is the so called Beta-Function of parameter $(\alpha+1)$ and $(\beta+1)$, that is the normalizing factor of the area under the PDF function. The trends of $p(a/a_M)$ and $P(a/a_M)$ are shown in Fig. 3 for different values of the shape parameters α and β.

To obtain the local failure probability distribution, it is necessary to define a local failure criterion. A penny-shaped crack of radius a, subjected to orthogonal tensile stress σ, is considered; the stress of fracture propagation is expressed by LEFM (Fig. 4). This criterion can be provided in a more convenient form, in terms of the maximum defect size a_M, that corresponds to the failure stress σ_u:

$$\frac{\sigma}{\sigma_u} = \sqrt{\frac{a_M}{a}}. \qquad (6)$$

In this way σ_u becomes the threshold stress beyond which there is no-probability of failure. The physical meaning of σ_u is thus identical to that of threshold strength proposed by Weibull (1939) in his statistical distribution. The failure criterion adopted in this formulation is characterized by a monotonically decreasing trend. This allows, Freudenthal (1968), to obtain a CDF for the local failure probability distribution $P_{f0}(\sigma/\sigma_u)$ by using the CDF of the defect size $P(a/a_M)$, and the criterion of (eq. 6), in the following form:

$$P_{f0}\left(\frac{\sigma}{\sigma_u}\right) = \begin{cases} 0 & ,\sigma \le \sigma_u \\ 1 - P\left(\frac{\sigma_u^2}{\sigma^2}\right) & ,\sigma \ge \sigma_u, \end{cases} \qquad (7)$$

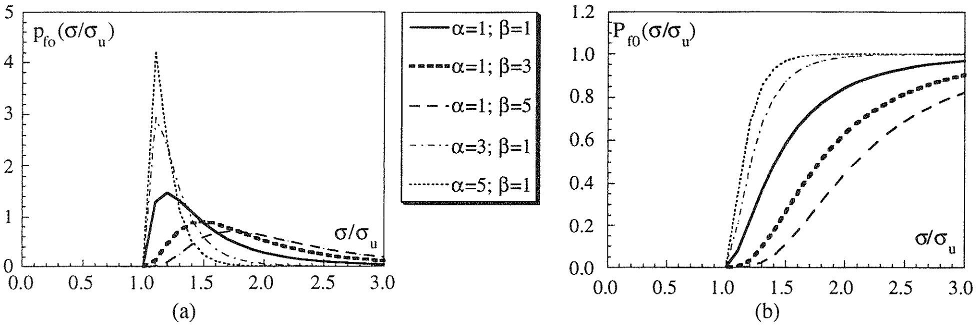


Fig. 5. Local failure probability for the truncated defect size model by varying the parameters α and β: (a) probability density; (b) cumulative distribution.

where σ is the critical stress associated with the incipient propagation condition of the defect of size a. Substituting eq. 5 into eq. 7, the local failure probability $P_{f0}\left(\frac{\sigma}{\sigma_u}\right)$ becomes:

$$P_{f0}\left(\frac{\sigma}{\sigma_u}\right) = \begin{cases} 0 & ,\sigma \leq \sigma_u \\ 1 - \dfrac{1}{B(\alpha+1,\beta+1)} \displaystyle\sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} \dfrac{1}{\alpha+i+1} \left(\dfrac{\sigma_u}{\sigma}\right)^{2(\alpha+i+1)} & ,\sigma \geq \sigma_u . \end{cases} \tag{8}$$

Deriving eq. 8 with respect to the macroscopic stress σ, the PDF of the local failure probability is obtained:

$$p_{f0}\left(\frac{\sigma}{\sigma_u}\right) = \begin{cases} 0 & ,\sigma \leq \sigma_u \\ \dfrac{2}{\sigma_u B(\alpha+1,\beta+1)} \displaystyle\sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} \left(\dfrac{\sigma_u}{\sigma}\right)^{2(\alpha+i+1)+1} & ,\sigma \geq \sigma_u . \end{cases} \tag{9}$$

The PDF and the CDF of the local failure probability are plotted in Fig. 5 for different values of the parameters α and β, as functions of the macroscopical stress σ normalized with respect to the limit tension σ_u. It is possible to observe how high values of parameter α are related to a PDF of the defect size accumulated at the largest value a_M. Therefore, the PDF of the local failure probability P_{f0} shows a small variance of the strength mean value, very close to σ_u. Consequentely, the microstructure presents a more ordered behaviour. On the other hand, high values of parameter β correspond to a defect distribution with small crack size predominance. In this case, the local failure probability is characterized by a large dispersion and the microstructural disorder increases.

Some hypotheses can make the analytical computation easier. It is considered that the interaction among the defects is negligible and that each volume element of the whole structure is made of the same material and subjected to the same direct tensile state of stress. Under this assumptions, the weakest link concept provides the probabilistic problem of the composition of n independent, equidistributed random variables:

$$P_{nx} = 1 - (1 - P_x)^n. \tag{10}$$

If V is the volume of the entire structure, V_0 the elementary volume on which the defect distribution $P(a)$ is defined, $n = V/V_0$ being the volume ratio, the global failure probability becomes:

$$P_n = \begin{cases} 0 & ,\sigma \leq \sigma_u \\ 1 - \left\{ \frac{1}{B(\alpha+1,\beta+1)} \sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} \frac{1}{\alpha+i+1} \left(\frac{\sigma_u}{\sigma}\right)^{2(\alpha+i+1)} \right\}^{\frac{V}{V_0}} & ,\sigma \geq \sigma_u. \end{cases} \tag{11}$$

Then, if bidimensional structural similarity is considered, the ratio V/V_0 becomes equal to d^2/d_0^2 and the global probability of failure takes the form:

$$P_n = \begin{cases} 0 & ,\sigma \leq \sigma_u \\ 1 - \left\{ \frac{1}{B(\alpha+1,\beta+1)} \sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} \frac{1}{\alpha+i+1} \left(\frac{\sigma_u}{\sigma}\right)^{2(\alpha+i+1)} \right\}^{\frac{d^2}{d_0^2}} & ,\sigma \geq \sigma_u, \end{cases} \tag{12}$$

where d is the structural dimension and d_0 is a characteristic length for the microstructure. Deriving eq. 12 with respect to the stress σ, the PDF of the global failure probability is obtained:

$$p_n = \begin{cases} 0 & ,\sigma \leq \sigma_u \\ \frac{2\frac{d^2}{d_0^2} \sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} \left(\frac{\sigma_u}{\sigma}\right)^{2(\alpha+i+1)+1}}{\sigma_u\, B^2(\alpha+1,\beta+1)} \left\{ \sum_{i=0}^{\beta} (-1)^i \binom{\beta}{i} \frac{1}{\alpha+i+1} \left(\frac{\sigma_u}{\sigma}\right)^{2(\alpha+i+1)} \right\}^{\frac{d^2}{d_0^2}-1} & ,\sigma \geq \sigma_u \end{cases} \tag{13}$$

The PDF and CDF of the global failure probability are plotted in Fig. 6, as functions of σ/σ_u, for different values of the volume ratio n. Increasing n, or the structural volume, the global failure probability is characterized by a smaller

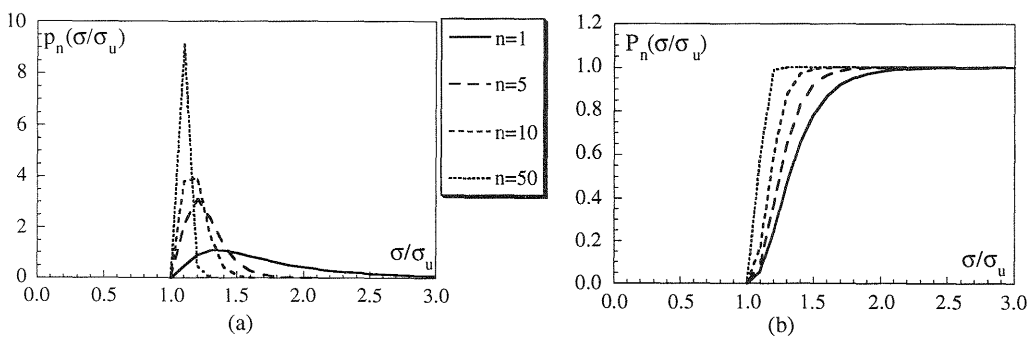


Fig. 6. Global failure probability for the truncated defect size model by varying the structural volume ratio n: (a) probability density; (b) cumulative distribution.

dispersion, while the mean value of the failure stress tends to σ_u. A very large structure tends to a deterministic behaviour with failure stress equal to σ_u. On the other hand, decreasing n, i.e. for smaller structural size, the dispersion increases, increasing in this way the failure stress scattering. This agrees with the physical evidence that the microstructural disorder is visible and influences the carrying capacity only at the small scales, whereas it vanishes for larger dimensions for which the structure appears as homogeneous. This model is thus suitable for describing the scale effect on the nominal strength for wide dimensional ranges. Eq. 13 in fact links the strength σ of a particular fractile P_n for a material characterized by a defect population with parameters a_M, α, β, and a characteristic length d_0 of the microstructure, with the structural dimension d. Due to the analytical complexity of the expression of P_n (eq. 12), it is not possible to write the relation between the strength and the structural dimension in a closed form. Therefore, the help of a numerical routine of function inversion is needed. The first step, in this case, must be the parametric analysis of the model in order to evaluate the influence of the different terms as well as their physical meaning.

4 Parametric analysis of the model

The results of the parametric analysis of the model are presented with the help of a numerical method of inversion. The output of the model is the strength of the structure as a function of the structural dimension and of the above mentioned parameters:

$$\sigma = \sigma(d; \alpha, \beta, a_M, d_0, P_n). \tag{14}$$

What has first been analyzed is the influence of the maximum theoretical defect size. This is equivalent to analyze the influence of σ_u, because the two values are deterministically linked by the local failure criterion (Fig. 7). Note that the validity range of the probabilistic model covers only structural dimensions greater than d_0,

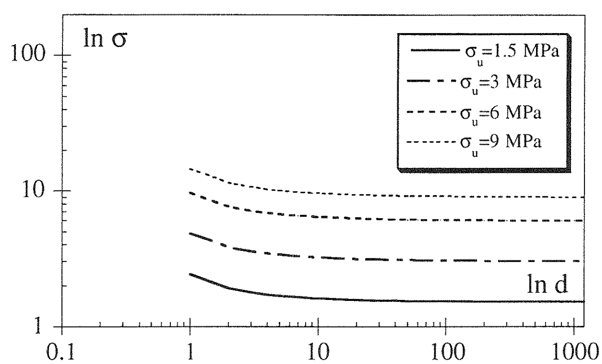


Fig. 7. Strength versus size, varying the parameters a_M or σ_u ($\alpha = 1$, $\beta = 2$, $d_0 = 1$, $P_n = 0.5$).

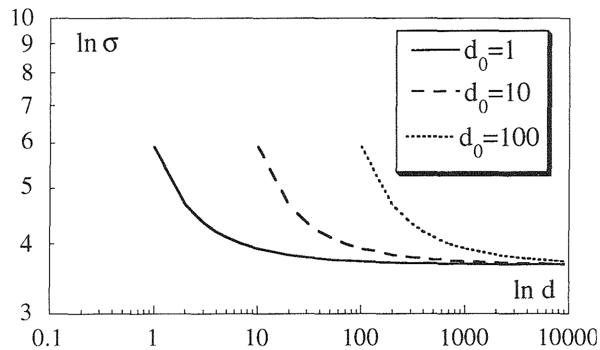


Fig. 8. Strength versus size, varying the parameter d_0 ($\alpha = 1$, $\beta = 2$, $\sigma_u = 3.67MPa$, $P_n = 0.5$).

or volumes V greater than V_0. V_0 is the minimum volume on which the dimensional defects distribution of eq. 4 is defined. It is obvious that the dimension of the maximum defect size a_M must be smaller than d_0.

It appears that the scale effect in the bilogarithmic diagram is non linear and shows an upward concavity. The parameter a_M, alias σ_u, is thus responsible for the asymptotic trend of strength for structural dimensions tending to infinity.

Structures with a larger maximum theorical defect a_M show a lower asymptotic strength. On the other hand, two structures with the same defect distribution show different asymptotic strengths only if they present different values of thoughness K_{IC}.

As was already said, the same material can exhibit a more or less evident size effect when different scale ranges are considered. The parameter that seems to be indicative of this phenomenon is the characteristic length d_0 (Fig. 8). Varying d_0, the scale effect curve shifts horizontally in the bilogarithmic diagram. A structure with a larger value of d_0 (i.e. volume element V_0 of the microstructure) could behave according to the disordered regime, whereas, for the same structural dimension d, a structure with a smaller characteristic lenght d_0 could be set in the nearly horizontal branch (Fig. 8).

The parametric analysis by varying α and β reveals how they can influence the

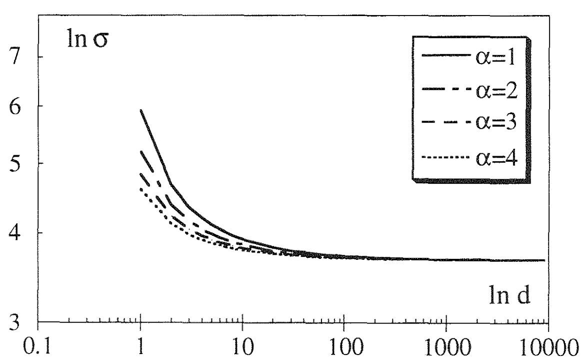


Fig. 9. Strength versus size, varying the parameter α ($d_0 = 1$, $\beta = 2$, $\sigma_u = 3.67MPa$, $P_n = 0.5$).

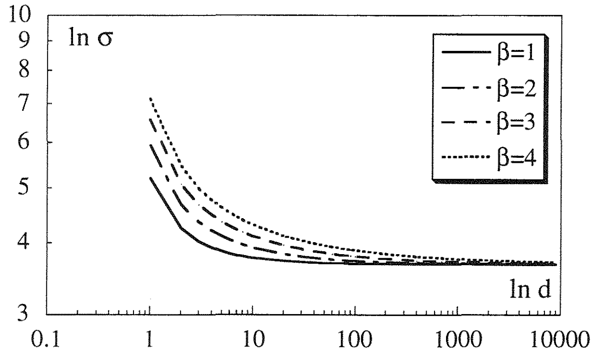


Fig. 10. Strength versus size, varying the parameter β ($\alpha = 1$, $d_0 = 1$, $\sigma_u = 3.67MPa$, $P_n = 0.5$).

slope of the size effect law. Increasing the parameter α the local strength, related to the structural size $d = d_0$, decreases (Fig. 9). On the other hand, increasing the parameter β the local strength increases (Fig. 10). As said before, a more ordered behaviour is related to larger values of the parameter α ($P(a/a_M)$ closer to the maximum size a_M). In the bilogarithmic plane σ versus d, this behaviour is represented by small difference between the local strength ($d = d_0$) and the asymptotic value σ_u. On the other hand, a larger dispersion of defect size distribution is obtained for larger values of parameter β. In this case, the slope of the left asymptote of the bilogarithmic σ versus d curve increases, representing an increment of microstructural disorder.

Eventually, the influence of the failure probability fractile P_n is considered. It appears that the size effect decreases by considering a small failure probability (fig. 11). This dependence on the fractile P_n is not shown by statistical models based on unbounded defects size distribution. Freudenthal (1968) shows how an unbounded statistical model can be characterized by a global failure cumulative distribution function that shifts horizontally to the left, without changing its shape, when the composition factor n grows. This corresponds to a scaling power laws represented,

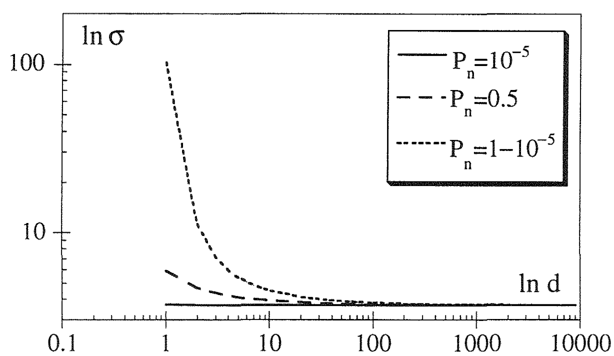


Fig. 11. Strength versus size, varying the fractile P_n ($\alpha = 1$, $\beta = 2$, $d_0 = 1$, $\sigma_u = 3.67MPa$).

in the bilogarithmic diagram, by a straight line, for any fractile considered. The limit of the unbounded model is due to the probability to find a defect size larger than the structural dimension.

It is worth noting that, when interpreting laboratory data, the best fractile to choose is $P_n = 0.5$, since it is related to the mean value of the nominal strength.

5 Comparisons with MFSL and experimental data

A comparison of the statistical size effect with the Multifractal Scaling Law will be proposed. It is obvious that such comparison is possible only in the dimensional range where both the models hold (for $d \geq d_0$). The MFSL can be expressed as:

$$\sigma_N = f_t \left(1 + \frac{l_{ch}}{d}\right)^{1/2} \tag{15}$$

where f_t is the asymptotic tensile strength for infinite structural dimensions, σ_N is the nominal tensile strength for a structure of size d, and l_{ch} is the characteristic length depending on the microstructure. The MFSL behaviour, in the bilogarithmic plane $\ln\sigma - \ln d$, shows two asymptots, with slope $-1/2$ in correspondence with the small scales, and slope zero for the large scales, respectively. The curve presents an upward concavity (fig.12).

The two approaches result to be in complete accordance. The analogies can be summarized in the following points.
(1) Both approaches predict an asymptotic nominal strength for large structural sizes. The statistical parameter σ_u and the MFSL parameter f_t certainly have the same physical meaning.
(2) In both cases, the size effect law, when plotted in the bilogarithmic plane $\ln\sigma - \ln d$, is nonlinear with upward concavity.

The bases of the MFSL and those of the truncated statistical model are very similar. They predict a transition from a disordered regime at the smaller scales, to an ordered regime at the larger scales. At the smaller scales, with the condition $d \leq d_0$, the truncated statistical model is characterized by a strongly strength dispersion. In this case, the strength value, for a given fractile (Fig. 6), decreases with

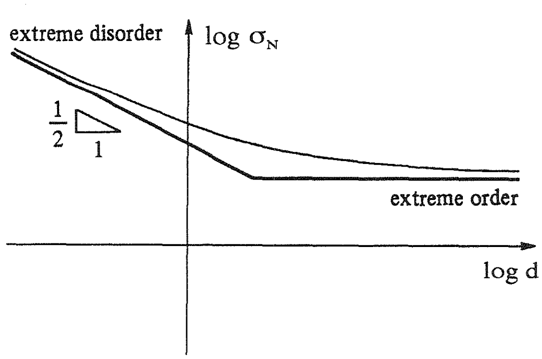


Fig. 12. Multifractal Scaling Law.

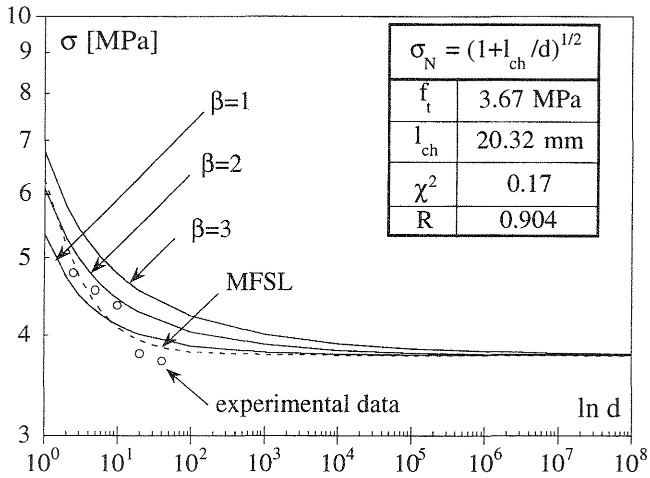


Fig. 13. Data fitting with the MFSL and the truncated statistical model.

increasing the structural size. On the other hand, at larger scales, the statistical composition of the local failure probability causes a flattening of the strength scale effect, and the distribution tends to the σ_u value with zero variance.

This trend is correlated with the MFSL, i.e. with the hypothesis of self-affinity for the material ligament. In other words, the physical dimension of the reacting section at the peak load can be identified by two different values of the fractal dimension: a local fractal dimension, in the limit of scales tending to zero, and a global fractal dimension, corresponding to the largest scales, strictly equal to the (integer) topological dimension. In consequence of this, the nominal tensile strength is constant for relatively large sizes, whereas it decreases with the size for relatively small sizes.

Missing a closed form expression (eq. 14) for the description of the nominal strength decrease, it is impossible to link explicitly the characteristic length d_0 with the corresponding l_{ch}, in spite of their evident identical meaning. It is possible to find particular values of the parameters that present a good correspondence with experimental data obtained from concrete tensile tests performed at the Politecnico di Torino, Carpinteri and Ferro (1994b, 1995). The best-fit parameters are shown in Fig. 13 ($\alpha = 1$, $\beta = 2$, $a_M = 216$ mm, $P_n = 0.5$ and $d_0 = 10$ mm).

7 Conclusions

A truncated statistical model for analyzing the size effect on nominal tensile strength of concrete structures has been developed. It allows the description of the nominal tensile strength variation when a large scale range is considered. The model provides an accurate description of the microstructural disorder, and a continuous transition of the tangential slope in the bilogarithmic diagram is obtained. In this way a continuous transition from order to disorder may be evidenced, by varying the values of the defect distribution parameters.

The defect size distribution of self-similarity is strictly related to the monofractal hypothesis of material ligament, where the disorder parameter N can be associated with the physical noninteger dimension of the reacting section. With these approaches, a size effect law characterized by a linear slope in the bilogarithmic diagram is obtained.

On the other hand, the truncated statistical model is in a good agreement with the results of the MFSL, giving a transition from a disordered regime to an ordered one, when the structural size increases. In conclusion, the truncated model represents in a better way the physical nature of the materials with respect to the self-similar statistical models, the maximum defect size being always smaller than the elementary volume dimension. On the contrary, unbounded statistical distributions provide a nonzero probability value to find a defect size larger than the structural dimension, in contrast with the physical evidence.

Acknowledgements

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