Fracture Mechanics of Concrete Structures,
Proceedings FRAMCOS-2, edited by Folker H. Wittmann,
AEDIFICATIO Publishers, D-79104 Freiburg (1995)

DAMAGE IN DISCRETE AND CONTINUUM MODELS

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Abstract
Failure in quasi-brittle materials occurs first progressively and then suddenly when micro cracks localise into a macro crack. From the continuum point of view, damage models are well suited to capture the essential features of such a behaviour. With the analysis of damage in lattices, we demonstrate in this paper that the pertinent variable which characterises damage is the variation of secant stiffness modulus. The analysis shows also that an internal length should appear in a continuum equivalent to a lattice of infinite size, due to spatial redistribution. Finally, a new damage model where the variation of the volume fraction of the material due to straining appears explicitly is proposed. This model incorporates an internal length which does not appear in the evolution law of damage, but in the field equations governing the variation of volume fraction.

1 Introduction

In phenomenological damage models, damage is very often understood as a degradation of the elastic stiffness of the material (see e.g. Lemaitre, 1992). The unrealistic features of damage localisation due to strain

softening are usually circumvented by the introduction of an internal length in the continuum description which scales the localisation process and controls the size of the damage / strain localisation zone (Pijaudier-Cabot and Bazant 1987, Pijaudier-Cabot and Benallal 1993, Sluys 1992). This paper is concerned with two aspects of such models:

The first aspect is the motivation for introducing such an internal length. It is based on the analysis of failure in lattices presenting an initial disorder. Infinite size lattices are equivalent to a continuum description. After having presented typical lattice responses, we analyse several quantities such as the distribution of cracking and its moment distribution. In particular, we show that the consistent variable, i.e. lattice size independent, which captures the effect of damage is the global conductance / stiffness of the model. In the lattice modelling, the internal length appears as a correlation length due to spatial redistribution and interactions during the failure process. The variation of this length with the system size is studied numerically.

Localisation is also characterised by a brutal variation of the volume fraction of the material. The volume fraction decreases very fast in the localisation zone to reach 0, that is a crack. In the second part, we examine a constitutive relation where the volume fraction of the material is also a cinematic variable. An existing elastic model is extended in order to describe progressive micro-cracking. The model is based upon the theory of elastic materials with voids. The volume fraction is related to the damage internal variable. Hence, the variation of the elastic stiffness due to micro cracking is coupled with the variation of volume fraction of the material. The properties of this model are quite comparable to that of a non local damage model.

2 Damage in a discrete system

We are studying here a simple discrete system which captures the essential ingredients of the behaviour of a disordered material. It is a regular two-dimensional lattice whose bonds are one dimensional (see Fig. 1-a). The lattice size is $L \times L$ where L is related to the total number of bonds $n = 2L^2$. Each bond behaves linearly up to an assigned threshold where brittle failure is reached. The model does not aim at describing a specific material. It ought to be the simplest model whose thermodynamic limit (lattice of infinite size) should be described by the damage theory. Yet, this description includes the essential ingredients: a two dimensional geometry, initial disorder, interactions and redistribution as the number of broken bonds increases.

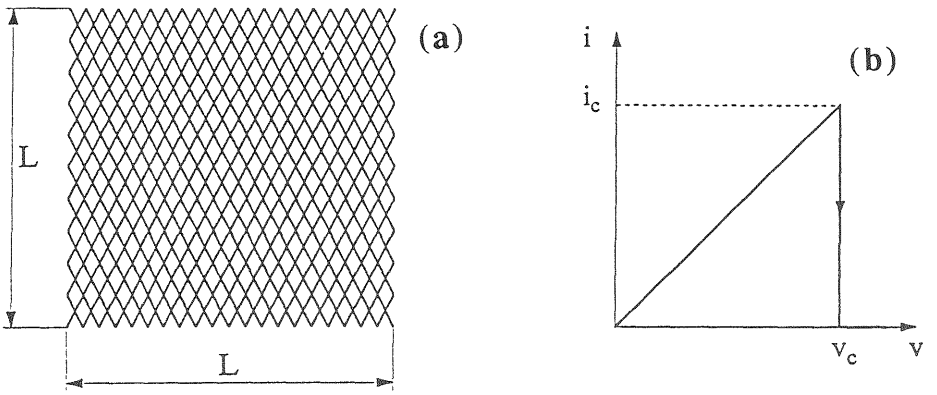

Fig. 1: (a) Lattice used for the analysis; (b) behaviour of one bond.

Instead of solving a mechanical problem, we use an electrical analogy which turns out to be strictly equivalent to the mechanical problem. The equations of equilibrium are similar but one dimensional instead of being two dimensional. The results can be interpreted in a mechanical fashion provided the current is replaced by the stress, the voltage by the strain and the conductance by the Young's modulus. The scaling properties of the mechanical problem and those of the electrical problem are indistinguishable (de Arcangelis and Herrmann, 1989). Figure 1-b shows the behaviour of one bond. The same conductance (stiffness) is assigned to all the bonds. The heterogeneity of the material properties is restricted to the variability of the maximum current at failure i_c (which is equivalent to the peak stress). We have chosen here a distribution which is constant between 0 and 1, hence it is representative of a large disorder which yields to diffuse damage which localises progressively. The boundary conditions at the limit of the lattice are periodic so that the behaviour of an infinite system is represented and boundary effects are avoided. A constant jump of voltage is applied along the two boundaries which are perpendicular to the direction y, and symmetry conditions are applied along the two other boundaries. At each stage of loading, a unit current I (load) is applied on the lattice. The current in each bond i is computed (solution of a linear algebraic system of $n/2$ equations according to Kirchhoff law) along with the overall conductance of the lattice. The computational algorithm removes one bond at a time. In order to determine the next bond to be broken at a given step, we look for the bond where the ratio i_c / i is minimum. This bond will fail when the current applied to the lattice is:

$$I_c = \min\left(\frac{i_c}{i}\right) \tag{1}$$

The voltage V_c and the overall conductance of the lattice are obtained from this critical current. Several computations with different random seeds have been performed and analysed in a statistical fashion. Figure 2 shows one of these plots for a lattice of size (32×32).

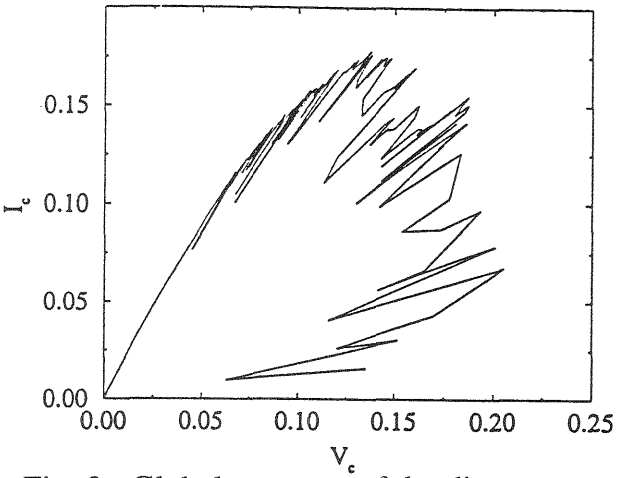

Fig. 2 : Global response of the discrete system.

2.1 Global properties of the discrete model

The global properties of the discrete model at any state of damage are, in this approach, directly connected to the local distribution of current $N(i)$. Instead of characterising the distribution itself, valuable information can be obtained with the analysis of the moments of the distribution of the current. The moment of order m is defined as:

$$M_m = \int i^m N(i) di \qquad (2)$$

Here, we will limit the analysis to moment of order up to 4. This assumption is equivalent to a truncature in a series development. These moments are of interest because of their physical meaning : the moment of order zero is the number of unbroken bonds. The first order moment is related to the average current, the second order moment is proportional to the overall conductance G:

$$M_2 = \int ri^2 N(i) di = 2GV^2 = 2G \qquad (3)$$

where r is the local resistance of the bond (unit resistance here) and V is the global voltage jump applied to the lattice, equal to one in our case.

Note that the fourth order moment is a measure of the dispersion of conductance.

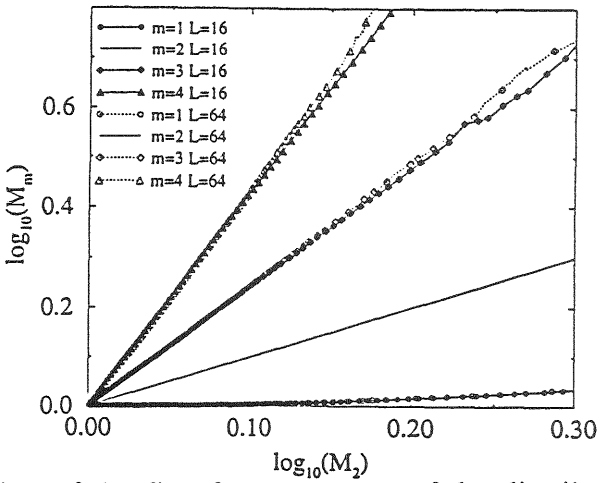

Fig. 3: Evolution of the first four moments of the distribution of current as a function of the overall conductance.

It is natural to investigate whether the number of broken bonds is a variable which characterises correctly the evolution of damage in the continuum sense, that is a degradation of the global conductance / stiffness. If the number of broken bonds q / L^2 (divided by the dimension of the lattice in order to have a quantity which is size independent) is the pertinent damage variable, the plots of the global conductance versus the non dimensional number of broken elements should be independent of the size of the discrete model. Delaplace et al. (1995) have shown that this is not true. Figure 3 shows a log-log plot of the moments as a function of the *second* order moment for different sizes of lattice. These plots do not depend on the size of the system, at least before the peak is reached. Damage can be defined as the variation of the overall conductance during failure. This variable describes well the distribution of the current (stress) in a *size independent* fashion.

2.2 Existence of a correlation length

Assume that there is a length denoted as ξ which defines the smallest size of the representative volume of the material (RVE). By construction, this length is smaller or equal to the size of the lattice. This quantity is a correlation length in a statistical sense. Above this size, the material can be regarded as homogeneous, i.e. without correlation. The local response of each bond in the discrete model is not independent for cells in the lattice bellow this size. In such cells, the response cannot be regarded as that of an homogeneous continuum. In order to exhibit the existence of a correlation length as the number of broken bonds

increases, one can analyse their distribution over the cell. This analysis is based on the variations of distance between two bonds in the lattice which are consecutively broken during the loading history according to the foregoing definitions: the distance between to neighbouring vertical bonds is 1 because there are L bonds on each column of the cell, and the distance between to neighbouring horizontal bonds is 0.5 because there are $2L$ bonds on each row of the cell.

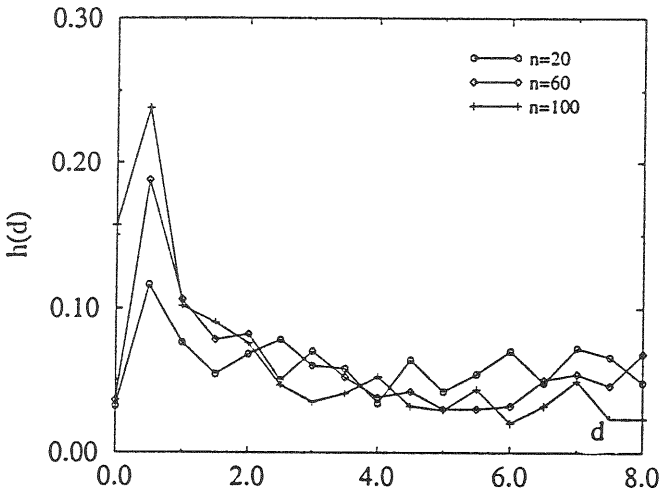

Fig. 4 : Histograms of the distribution of horizontal distance.

Figure 4 shows for systems of size 16x16 the histograms of the distribution of horizontal distance between two consecutive broken bonds denoted as $h(d)$ at the beginning of damage (20 broken bonds), near the peak (60 broken bonds) and near failure (100 broken bonds). In the absence of a correlation length, these histograms should be a set of horizontal lines, which means that the next broken bond can found anywhere in the lattice. These histograms can be approximated at least up to the peak: the distribution of the broken links is assumed to follow a power law function of the distance up to the unknown distance ξ and then it is an horizontal line. The length ξ corresponds then exactly to the smallest size of the RVE:

$$h(d) \approx d^{-\tau} \text{ for } d \leq \xi,\ h(d) = h^* \text{ for } d \in \left[\xi, \frac{L}{2}\right] \qquad (4)$$

The value of exponent τ is 0.67 for the normalised distributions shown on figure 4. The distance ξ is also the length at which the distribution changes from diffuse damage (constant distribution) to a progressive localisation of damage. Hence, it can be considered as a measure of the size of the zone in which damage localises. The first order moment of

the distribution $h(d)$ can be obtained numerically. Substitution of Eq. (4) in the expression of the first order moment yields an equation where the only remaining unknown is the correlation length.

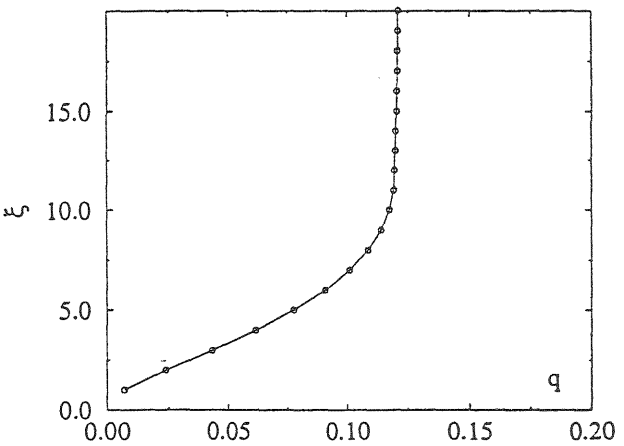

Fig. 5 : Internal, correlation length vs. number of broken bonds.

Figure 5 shows the evolution of this correlation length. It increases with increasing damage.

3 Damage in elastic continua with voids

Two characteristics should be kept in mind when devising a continuum model for quasi-brittle materials: growing voids and defects can be captured with the variation of secant stiffness of the material and an internal length should be present in the model. In the past, several damage models have been proposed along this line (see e.g. Bazant and Pijaudier-Cabot 1987, Frémond and Nedjar 1993). We shall concentrate here on an extension of the model devised for elastic materials with voids due to Cowin and Nunziato (1983).

This models introduces two cinematic variables, the displacement $u(x,t)$ and the change in volume fraction of solid material $\phi(x,t)$. Each variable possesses its own governing equation (here in the absence of body forces):

$$\begin{aligned} \rho \ddot{u}_i(x,t) &= \sigma_{ij,j} \\ \rho k \ddot{\phi} &= h_{i,i} + g \end{aligned} \tag{5}$$

where u_i is the ith component of the displacement, ρ is the current mass density of the material, h_i is the equilibrated stress vector, k is the

equilibrated inertia, g is the equilibrated body force, and σ is the stress. Variable h_i is the force associated to the variation of volume fraction and can be related to classical elastic singularities such as a centre of dilation (void) or a couple of opposite forces applied at the same point or (crack). Note that essential and natural boundary conditions exists for each governing equation and that Eq. (5-b) is quite similar to the governing equation for the variation of damage in the model by Frémond and Nedjar (1993). The elastic (free) energy reads:

$$W = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \beta \delta_{ij} \varepsilon_{ij} \phi + \frac{1}{2} \alpha \phi_{,i} \phi_{,j} + \frac{1}{2} \xi \phi^2 \tag{6}$$

where C_{ijkl} are the stiffness coefficients and (α, β, ξ) are material parameters. The thermodynamic forces associated to the internal variables $(\varepsilon_{ij}, \phi, \phi_{,i})$ are defined as:

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}, g = -\frac{\partial W}{\partial \phi}, h_i = \frac{\partial W}{\partial \phi_{,i}} \tag{7}$$

As pointed out by Cowin and Nunziato (1983), the stiffness coefficients and the material parameters should depend on the reference (i.e. when the material is free of loads) volume fraction. Therefore the influence of damage, assumed here to represent void growth and cracking, can be introduced by setting these parameters as functions of an internal scalar damage variable (for simplicity) denoted as D. The new free energy reads:

$$W = \frac{1}{2}(1-D) C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \beta(D) \delta_{ij} \varepsilon_{ij} \phi + \frac{1}{2} \alpha(D) \phi_{,i} \phi_{,j} \delta_{ij} + \frac{1}{2} \xi(D) \phi^2 \tag{8}$$

with the additional associated variable

$$Y = -\frac{\partial W}{\partial D} \tag{9}$$

Note that the influence of damage on the stiffness coefficients is similar to what can be found in standard models (Lemaitre, 1992). For the sake of simplicity, we will consider that the other material parameters in Eq. (8) are not function of damage in the present analysis.

The growth of damage is similar to the usual evolution law. We define a loading function and an associated evolution law:

$$f(Y,D) = \int_0^Y F(z)dz - D, \dot{D} = \delta \frac{\partial f}{\partial Y} \tag{10}$$

with the classical Kuhn-Tucker conditions. δ is the damage multiplier and F is an experimentally determined hardening/softening function.

Let us consider now the one dimensional version of such a model and investigate the occurrence of bifurcation from a homogeneous state of deformation, variation of volume fraction, and damage, denoted as $(\varepsilon^0, \phi^0, D^0)$. The rate constitutive relations (i.e. for infinitesimal perturbations about the initial state) are obtained from Eqs. (7-10):

$$\begin{cases} \dot{\sigma} = (1-D^0)E\dot{\varepsilon} - (E\varepsilon^0)^2 F(Y^0)\dot{\varepsilon} + \beta\dot{\phi} \\ \dot{h} = \alpha\dot{\phi}_{,x} \\ \dot{g} = -\xi\dot{\phi} - \beta\dot{\varepsilon} \end{cases} \tag{11}$$

Let us restrict the analysis to static's and omit the inertia terms in Eq. (5). Substitution of Eqs. (11) in the governing equations yields:

$$\begin{cases} \left(E(1-D^0) - (E\varepsilon^0)^2 F(Y^0)\right)\dot{\varepsilon}_{,x} + \beta\dot{\phi}_{,x} = 0 \\ \alpha\dot{\phi}_{,xx} = \beta\dot{\varepsilon} + \xi\dot{\phi} \end{cases} \tag{12}$$

Substitution of Eq. (12-b) into Eq. (12-a) yields the differential equation governing the variation of volume fraction:

$$\alpha\dot{\phi}_{,xxx} + \left(\frac{\beta^2}{\left(E(1-D^0) - (E\varepsilon^0)^2 F(Y^0)\right)} - \xi \right)\dot{\phi}_{,x} = 0 \tag{13}$$

This linear differential equation admits harmonic solutions if its coefficients have the same sign. Assume that coefficients (β, ξ) are positive. The meaning of this assumption can be interpreted by considering cases where damage is constant and the variation of volume fraction is homogeneous. Eqs. (5-b,6,7) yield:

$$\phi = -\frac{\beta}{\xi}\varepsilon_{kk} = -\frac{\beta}{\xi}\frac{\Delta V}{V} \tag{14}$$

Hence the volume fraction diminishes as the total volume of material increases due to straining.

The quotient in the fraction of the second coefficient of Eq. (13) is negative upon softening. It follows that α must be negative if harmonic functions are to be found for the variation of volume fraction and strain distributions at the bifurcation (with the restriction that the free energy Eq. (8) must be positive definite). Exponential solutions cannot comply with the boundary conditions since the spatial derivative of the volume fraction must be continuous at the loading/unloading interface. Same as for the non local damage model (Pijaudier-Cabot and Benallal, 1993) only bifurcation into harmonic modes of non zero wave length is admissible. It is important to remark also that an internal length appears in the coefficient α. A dimensional equation shows that this coefficient is homogeneous to the square of a length.

As a closing remark, let us point out that this model complies with the requirements derived for the analysis of discrete systems and that a plasticity model could be devised as well in the same spirit where the variation of volume fraction could be interpreted as non linear dilatancy effects.

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