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FRACTURE OF CONCRETE
A NEW APPROACH TO PREDICT CRACK PROPAGATION

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Abstract

A newly developed crack growth model, Nielsen (1990), based on an energy balance criterion is presented in the paper. The model leads to a first order differential equation which can predict crack propagation in mode I, and determine the strength and deformations for any loading condition. The model has recently been used to predict crack growth in metals subjected to dynamic loading, Hansen (1994a).

In this paper the model is compared with three point bending tests on plain concrete beams, where several parameters have been varied: the beam dimensions, the strength and the aggregate size.

1 Introduction

A newly developed fracture mechanical model, Nielsen (1990), based on an energy balance criterion is presented in the paper. The model leads to a first order differential equation which can predict crack propagation, and determine the strength and deformations for any loading condition. The purpose of this paper is to determine the load-deflection relationship for plain concrete beams in three point bending using this model.

The advantage of the model presented in this paper compared to other models of fracture mechanics for concrete is that the energy calculation is based on simple linear elastic calculations, which demand very little computation time. The non-linear behavior is taken into account, using a crack length correction based on the Irwin concept, Irwin (1960).

Determining the total elastic energy W in a cracked specimen using linear elasticity gives results close to the correct values, due to the fact that the stress concentration effects in the process zone close to the crack tip are negligible. Introducing an effective crack length $a_{eff} = a + l_e$, where l_e is Irwins crack length correction, Irwin (1960), which represent a part of the process zone, it is possible to take the non-linear behavior into account.

2 The energy balance crack propagation formula

In the following the energy balance crack propagation formula (ECP) will shortly be presented. For further details the reader is referred to, Nielsen (1990). ECP is based on an energy criterion. Energy criteria were introduced in fracture mechanics by Griffith (1921). For a displacement controlled test, where the crack length a and the deflection u are the independent variables, the energy balance equation can be written, Nielsen (1990):

$$\frac{\partial W}{\partial a}da + G_F b da = 0 \tag{1}$$

Here W is the elastic energy, G_F is the fracture energy, defined as the energy needed to propagate the crack a unit area and b is the thickness of the specimen. Taking the correction l_e of the crack length into account we get:

$$\frac{\partial W}{\partial a}da + \frac{\partial W}{\partial a}dl_e + G_F b da = 0 \tag{2}$$

which can be rearranged to:

$$da = \frac{-\frac{\partial W}{\partial a}dl_e}{G_F b + \frac{\partial W}{\partial a}} \tag{3}$$

Since the crack length correction l_e depends on a as well as u we have:

$$dl_e = \frac{\partial l_e}{\partial u}du + \frac{\partial l_e}{\partial a}da \tag{4}$$

Inserting this into (3) and making a few rearrangements we get:

$$\frac{da}{du} = \frac{-\frac{\partial W}{\partial a}\frac{\partial l_e}{\partial u}}{G_F b + \frac{\partial W}{\partial a}(1 + \frac{\partial l_e}{\partial a})} \qquad (5)$$

The derivatives $\partial W/\partial a$ should be taken at $a+l_e$, while $\partial l_e/\partial u$ and $\partial l_e/\partial a$ may be taken at a. This expression is called the energy balance crack propagation formula. In the symmetrical case, meaning a crack with two tips and the crack length 2a, G_F in the formula must be replaced by $2G_F$, W still being the total elastic energy.

3 Numerical determination of crack growth

As described earlier the crack propagation formula is based on an energy balance criterion. It is therefore necessary to determine the elastic energy in the actual cracked body (beam,disk etc.) for arbitrary values of the load.

The elastic strain energy W can be determined by means of a finite element calculation. In this way W will be expressed as a function of the force P or the displacement u and the crack length a.

In the case where one wishes to determine the load-deflection curve of a concrete specimen, it is convenient to express the energy W by the displacement u and a.

It is common knowledge that the elastic strain energy is proportional to the square of the displacement and proportional to the modulus of elasticity. This means if we have determined W in the case of constant displacement and constant modulus of elasticity for any crack length, then we can express the energy as:

$$W(u,a) = \left(\frac{u}{u_0}\right)^2 \cdot \left(\frac{E}{E_0}\right) \cdot W_{const\ E,u}(a) \qquad (6)$$

The problem is now reduced to determine the elastic energy of the actual cracked body as a function of the crack length - only depending on the geometry. Very simple finite element models can be used to determine this function. We only have to calculate a number of models with similar geometry, except for the length of the crack, subject all these models to a constant displacement and determine the force P. The elastic strain energy can then be determined as a function of P and u. In the case of only one concentrated load the elastic strain energy is given by:

$$W = \tfrac{1}{2} \cdot P \cdot u \qquad (7)$$

When the elastic strain energy is determined for each crack length, a simple polynomial fit may be used to determine the elastic strain energy for constant displacement at any crack length. By increasing the crack length, the energy curve (for constant displacement) decreases as shown in figure 1 in the case of three point bending of notched beams with rectangular section. W_0 is the strain energy related to a non cracked beam. From this curve the change in the elastic energy $\partial W/\partial a$ can be determined for any value of u and a using formula (6).

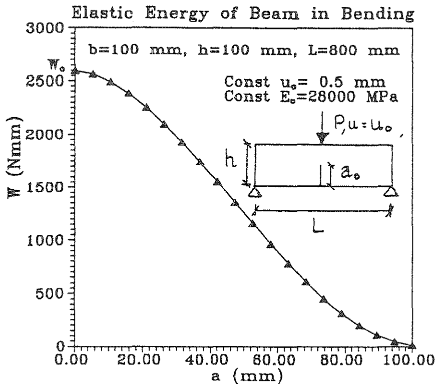

Fig. 1. Elastic energy W for three point bending under constant displacement

Taking into account the non-linear effects we introduce, as mentioned, the effective crack length $a_{eff} = a + l_e$. The correction length l_e is determined on the basis of Irwins concept, Irwin (1960). Some energy considerations presented in, Nielsen (1990), leads to the formula (8).

$$l_e = 0.4 \cdot \frac{K_I^2}{\pi f_t^2} \qquad (8)$$

Here the stress intensity factor K_I is determined by formula (9).

$$K_I^2 = -\frac{\partial W}{\partial a}\frac{E}{b} \qquad (9)$$

The correction length l_e is increasing for increasing crack length a, meaning that for a certain value of the crack length, the process zone a_p, which approximately is $a_p = 2.5 \cdot l_e$, see Nielsen (1990), will reach the top of the beam. At that point the calculation must stop, and therefore the theory does not predict fully the descending part, as will be shown in the next section.

To calculate the load-deflection curve using the crack propagation formula, several parameters have to be known. These are wellknown geometrical and mechanical parameters which can be determined by simple statical tests. The geometrical parameters are dimensions of the beam and the start notch length, the mechanical parameters are the modulus of elasticity E, the tensile stress f_t and the fracture energy G_F.

When a load-deflection curve has been calculated it is possible to check the energy balance by calculating the area ΔA in figure 2. We have:

$$G_F = \frac{\text{external energy supplied}}{\text{area of cracked surface}} = \frac{\Delta A}{b \cdot \Delta a} \qquad (10)$$

where b is the thickness of the beam, and Δa is the cracked surface, i.e. the total effective crack length minus the correction length l_e and the start notch length a_o:

$$\Delta a = a_{eff} - l_e - a_o = a - a_o \tag{11}$$

Using a numerical integration formula the area ΔA may be determined. Δa is found by solving numerically the equation (5).

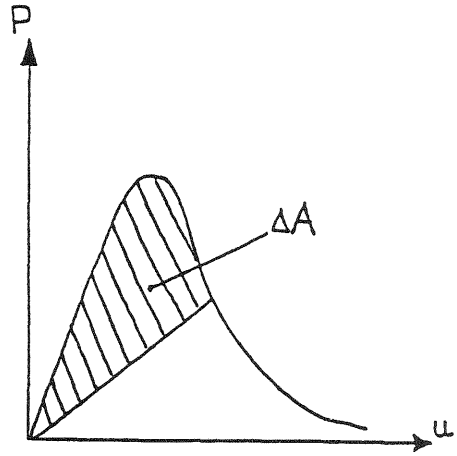

Fig. 2. Definition of the area ΔA from the load-deflection curve

Consider as an example the following standard values of the parameters: E=30000 MPa, f_t=2 MPa, a_o=50 mm, $b{\cdot}h{\cdot}L$= 100·100·800 mm^3 and let's separately vary the values of G_F, f_t, E and a_o. The results are shown in table 1. It's observed that the results show good accordance except for a small numerical error (which can be eliminated by decreasing the increment step).

Table 1. Verification of energy balance

	ΔA (Nmm)	Δa (mm)	$G_F=\Delta A/b{\cdot}\Delta a$	G_F(theory
standard	16.30	1.64	0.0993	0.1
G_F=0.05N/mm	17.69	3.58	0.0494	0.05
f_t=4 MPa	76.50	7.71	0.0992	0.1
E=40000 MPa	12.78	1.29	0.0989	0.1
a_o=25 mm	57.91	5.86	0.0989	0.1

Since the theory determines the load carrying capacity of any structure under the conditions described, it will be able to predict the size effect, i.e. it will be able to give the load carrying capacity as a function of the absolute value of any geometrical parameter D characterizing the size of the structure. As a measure of brittleness we take the common value:

$$B = \frac{f_t^2 D}{G_F E} \tag{12}$$

In our case we put D = h. Formula (12) reflects the fact that the larger the strength f_t and the size D the more brittle structure. The smaller the value of G_F or E the more brittle structure. Large values of B belong to perfectly brittle structures. For a perfectly brittle structure the load carrying capacity may be determined by Linear Elastic Fracture Mechanics (LEFM), i.e. by setting:

$$K_I = K_{IC} = \sqrt{G_F E} \tag{13}$$

where K_{IC} is the critical value of the stress intensity factor, the fracture thoughness. If we solve the crack propagation formula for increasing values of B by decreasing G_F, the load carrying capacity P_{peak} will be decreasing. The value of K_I^2/E at the peak should according to (13) approach the G_F value assumed in the calculation, if the results approach the load carrying capacity of LEFM. The result of such a calculation is shown in table 2, and we observe that the value of K_I^2/E approaches the value of G_F for increasing brittleness. The load carrying capacity has been given in dimensionless form as the Navier stress σ_o along the depth h-a_o devided by f_t, i.e:

$$\frac{\sigma_o}{f_t} = \frac{3}{2} \cdot \frac{P_{peak} L}{b(h-a_o)^2} \cdot \frac{1}{f_t} \tag{14}$$

Table 2. Load carrying capacity as a function of G_F

	G_F (N/mm)	E (MPa)	f_t (MPa)	a_o (mm)	b·h·L (mm)	
Standard:	varies	30000	2	50	100·100·800	
No	**G_F (N/mm)**	**B**	**P_{peak} (N)**	**σ_o/f_t**	**K_I^2/E**	**l_e (mm)**
g1	10000	1.33E-6	529.7	1.27	4.3E-3	11
g2	100	1.33E-4	529.7	1.27	4.3E-3	11
g3	1	1.33E-2	528.3	1.26	4.2E-3	11
g4	0.5	0.0267	526.9	1.26	4.2E-3	11
g5	0.2	0.0667	522.8	1.25	4.1E-3	10
g6	0.1	0.1333	516.1	1.24	4.0E-3	9
g7	50.0E-3	0.2667	503.4	1.21	3.8E-3	9
g8	10.0E-3	1.333	426.0	1.02	2.8E-3	6
g9	10.0E-4	13.33	209.9	0.50	6.7E-4	1
g10	10.0E-5	133.3	73.55	0.18	8.0E-5	0.1

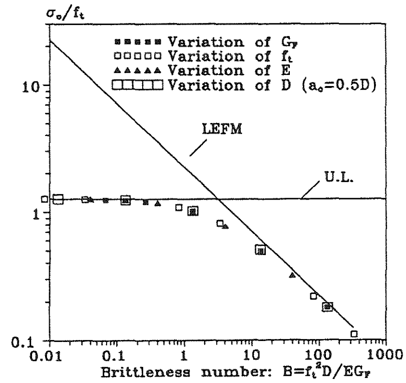

Fig. 3. Failure load versus Brittleness Number

The results are also depicted in figure 3 (points marked "variation of G_F"). It appears that when increasing the brittleness number B the load carrying capacity approaches the results given by LEFM, i.e. the results for a perfectly brittle structure. In the other end for small values of B, the result is expected to approach the load carrying capacity of an almost perfectly plastic structure.
For a perfectly unnotched plastic beam with rectangular section in bending with tensile strength f_t and infinite compressive strength f_c the theoretical upper limit (U.L.) equals $\sigma_o/f_t=3$. Having a finite compressive stress f_c and if the ratio between f_t and f_c is put to $\alpha=f_t/f_c=0.1$ we get the theoretical upper limit $\sigma_o/f_t=2.73$, see Olsen, P. (1994d).

For decreasing start notch length the results of the theory will approach a limiting value close to this upper plastic solution as shown in table 3. In our case, with a start notch depth being half the full depth h, the limiting value is reduced. As we observe in table 2 the value is found to be $\sigma_o/f_t=1.27$, in this case.

Table 3. The relative stress approaches the upper plastic limit for an unnotched beam when the notch length is decreased

G_F=0.1N/mm, f_t=2MPa, E=30000MPa, B=0.133			
a_o (mm)	P_{peak} (N)	σ_o/f_t	l_e (mm)
50	516.4	1.24	10
25	1327	1.42	14
10	2302	1.71	15
5	2905	1.93	13
1	3806	2.33	7
0.001	4306	2.58	6
0.00001	4307	2.58	6

In figure 3 also results found by varying the other parameters determining the brittleness number B have been shown. It appears that the results are lying on the same curve as before as they should.

The theory takes the size effect into account by the variation of the crack length correction l_e with K_I. If l_e is large (relative to the beam height) the material behaves ductile, in agreement with what we observe in table 2, where l_e is large for small values of the brittleness number.

4 Comparison with test results

In this section the new theory of crack propagation will be compared with notched three point bending tests. The three point bending tests were performed at the Department of Structural Engineering, Technical University of Denmark, see Olsen, D. (1994b) and (1994c).

For the purpose of verifying the crack propagation formula three different test series have been carried out. Several parameters were varied. Four different compression strength levels were used (f_c = 30, 50, 70 and 100 MPa). Further the ratio of the start notch and the depth of the beam a_0/h was varied being 0.1, 0.25 and 0.5. Finally the aggregate size was varied for each strength level. The maximum aggregate size d_{max} was: 0 mm, 4 mm, 8 mm and 16 mm.

As described in section 3 the theory is not able to predict the whole load-deflection curve, due to the fact that when the process zone reaches the top of the beam the assumptions used fail to be valid. Therefore the calculation is only carried out for a part of the load-deflection curve. To determine the load-deflection curve using the crack propagation theory, the following parameters have to be known: the fracture energy G_F, the tensile strength f_t and the modulus of elasticity E. As far as possible they should be determined independently of the bending test results.

The fracture energy G_F is in, Olsen, D. (1994b), determined using the three point bending method in accordance with the recommedations of the RILEM Technical Commity 50-FMC, Rilem (1985b). In this method the measured load-deflection curve does not give the total amount of energy consumed, due to the fact that the energy is not only supplied from the applied force but also from the netweight of the beam, which has to be taken into account when determining G_F.

This is done using the method suggested by Hillerborg (1985a), putting the total energy equal to the area under the measured load-deflection curve plus the secondary part from the netweight. G_F used in the theoretical calculations will be determined on the basis of formula (10), having $\Delta a = h - a_0$ and ΔA being the full area under the measured load-deflection curves. The calculated load-deflection curves have been determined for a concentrated load only, disregarding the netweight of the beam. These curves are compared with the measured load-deflection curves, the load meaning the applied concentrated load and the deflection meaning the deflection measured in the test having zero value when the beam is acted upon only by its own weight.

The splitting strength f_{sp} was determined for each concrete mix using standard procedures, see Olsen, D. (1994b). The tensile strength f_t used in the theory is put equal to $f_t = 1.25 \cdot f_{sp}$. This value is close to the flexural modulus f_b found in bending tests, which are higher than the uniaxial tensile strength f_u and the splitting strength f_{sp}. This problem will be discussed further in forthcoming papers.

The modulus of elasticity E was determined for all the concrete mixes by separate cylinder tests using standard procedures, see Olsen, D. (1994b).
The parameters used in the calculations are listed in table 4 - 6.

Table 4. Parameters used in the calculations shown in figure 4, d_{max} = 16 mm for all specimens

	E (MPa)	G_F (N/m)	a_o (mm)	f_{sp} (MPa)	$f_t=1.25 \cdot f_{sp}$
A_303N	25160	81.1	50	2.90	3.63
A_503N	32950	64.0	50	3.76	4.70
A_703	35300	85.8	50	4.82	6.03
A_1003	42210	95.7	50	5.49	6.86

Table 5. Parameters used in the calculations shown in figure 5. d_{max} = 16 mm for all specimens

	E (MPa)	G_F (N/m)	a_o (mm)	f_{sp} (MPa)	$f_t=1.25 \cdot f_{sp}$
A_501	33640	113.2	10	3.87	4.85
A_502	32490	106.2	25	3.72	4.65
A_503N	32950	64.0	50	3.76	4.70

Table 6. Parameters used in the calculations shown in figure 6. a_o = 50 mm for all specimens.

	E (MPa)	G_F (N/m)	d_{max} (mm)	f_{sp} (MPa)	$f_t=1.25 \cdot f_{sp}$
C_301	10450	3.7	0	2.00*	2.50
C_302	22120	18.0	4	2.27	2.84
C_303	27460	32.7	8	2.51	3.14
A_303N	25160	81.1	16	2.90	3.63

* The splitting strength for this test series was not measured. The value is based on interpolation from the other splitting strength measurements.

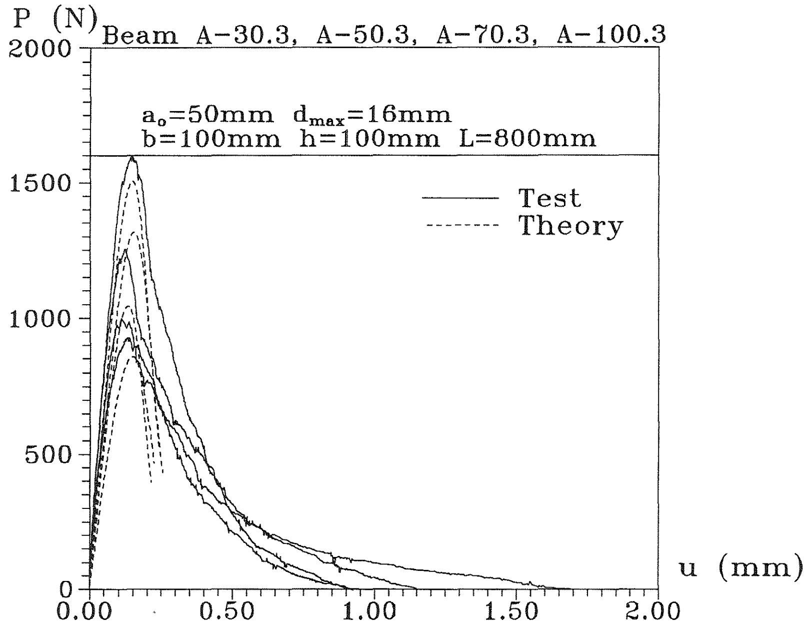

Fig. 4. Variation of Strength

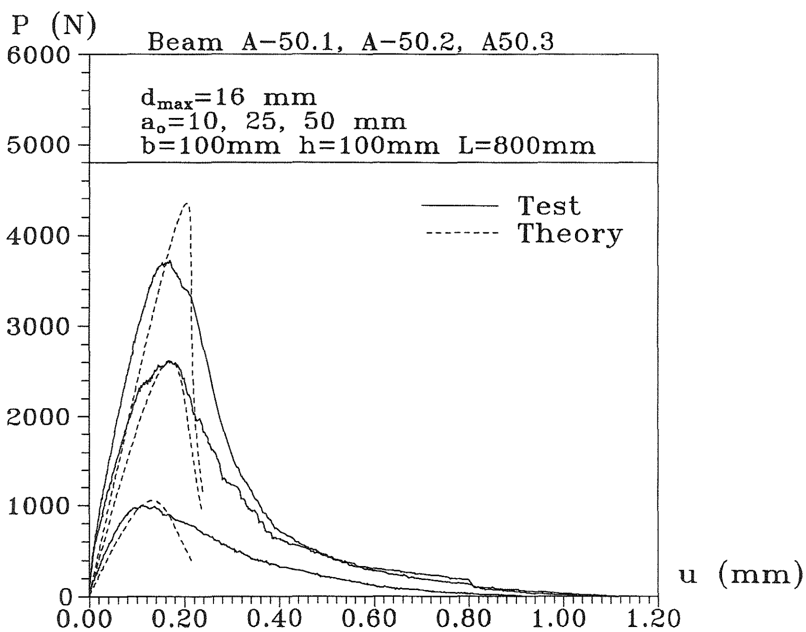

Fig. 5. Variation of notch length

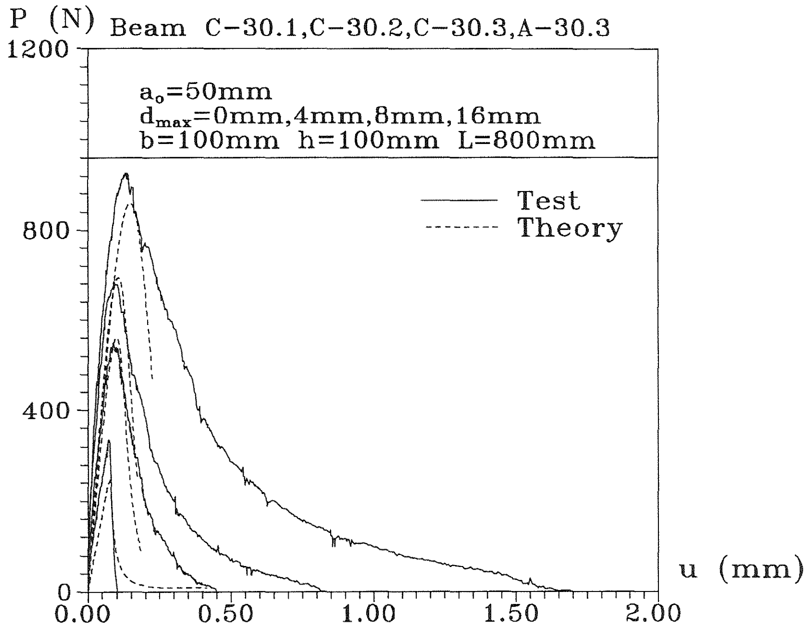

Fig. 6. Variation of aggregate size

Figures 4-6 show the measured load-deflection curves and the ones calculated by the crack propagation formula. The aggrement is in most cases found to be excellent.

5 Conclusion

The purpose of this paper has been to present a theoretical crack propagation formula and compare it to three point bending test results of plain concrete beams.

The main conclusions are that the new crack propagation theory presented is able to predict the peak value and some part of the descending part of the load deflection curve excellently in most of the cases, both for different strength levels, different ratios of start notch length and depth (a_0/h) and finally for different kinds of concrete materials (aggregate variation).

The theory is also able to take size effects into account, and it has been shown that the theory gives results approaching the plastic limit of the load carrying capacity for an unnotched beam.

One of the main advantages of the theory is that the theory needs very small computation times to reach the results, contrary to other methods of fracture mechanics.

6 Nomenclature

a	Crack Length	E, E_o	Modulus of Elasticity
a_{eff}	Effective Crack Length	f_t	Tensile Strength
a_o	Start Notch Length	f_b	Flexural Modulus
l_e	Crack Length Correction	f_u	Uniaxial Tensile Strength
a_p	Process Zone Length	f_{sp}	Splitting Strength
u, u_o	Displacement	W	Elastic Strain Energy
b	Width	K_I	Stress Intensity Factor
h	Depth	G_F	Fracture Energy
L	Length	B	Brittleness Number
D	Characteristic Size	σ_o	Stress
A	Area	P	Load

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