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DISCONTINUOUS DEFORMATION ANALYSIS FRAMEWORK FOR MODELLING CONCRETE FRACTURE

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Abstract

Some possibilities and limitations of the discontinuous deformation analysis in modelling concrete fracture are illustrated on several simple case studies, in particular in tracing the transition from a continuum to discontinuum. Computational issues involved are briefly reviewed and the recent trend in recasting the original formulation in its more general form as a manifold method is noted. Illustrative examples include the Kitoh plain concrete beam and the RILEM pull-out benchmark problem.

Key words: case studies, Kitoh plain concrete beam, RILEM pull-net benchmark, discontinuous deformation analysis DDA

1 Introduction

Discontinuous modelling frameworks have become increasingly utilised in concrete fracture simulations, including the discrete element method, rigid block method, lattice modelling and the discontinuous deformation analysis. Originating in the field of rock mechanics, the DDA analysis appeared initially (Shi1988) as an efficient framework of modelling jointed rock as a deformable blocky media.

The DDA method has recently been reformulated as a subset of a more generalised framework, and the procedure is seen as an alternative approach along with a number of approximation procedures suitable for modelling discontinuous media. The most recent generalisation is the Manifold Method (Shi 1997), which advocates very similar ideas as the ones used in the *meshless* methods (Belytschko et al 1994) and in the method of *moving least squares* (Nayroles 1992), where the development can traced back to early irregular finite difference schemes. Similar to the meshless methods, the manifold method identifies the cover displacement function and the cover weighting function, where the geometry of the actual blocks is utilised for numerical integration purposes. The treatment of discontinuities is envisaged in the same way as with meshless methods, i.e. by introducing the concept of effective cover regions, where there is a need for n independent covers if a cover intersects n disconnected domains.

The commonality of the two approaches, where the meshless methods stem from a continuum side and the manifold method from the discontinuum end of the spectrum, indicates a possibility of a more rigorous treatment in modelling of progressive discontinuities in quasi brittle materials.

Computationally, the original DDA method is effectively as an alternative way of introducing solid deformability into the discrete element framework, where block sliding and separation was considered along predetermined discontinuity planes. The original formulation was restricted to simply deformable blocks (constant strain state over the entire block of arbitrary shape in 2D), and the displacement field for each block was described by the three displacement components of the block centroid, augmented by the displacement field corresponding to the constant strain state, denoted by the block deformation vector $\mathbf{D}_i$. Components of the stiffness matrix and the load vector are obtained by the usual process of the minimisation of potential energy, and the distinction is made between the contribution to the potential energy of the whole system arising from the internal strain energy of the block itself and the potential energy associated with all contact constraints present.

Improved model deformability is achieved by either increasing the number of block deformation variables (higher order DDA, where higher order strain fields are assumed for blocks of arbitrary shapes), or by the so called sub-block concept, in which a block is subdivided into a set of simply deformable sub-blocks.

Minimisation of potential energy leads to a formation of a familiar set of equations with n mixed degrees of freedom for every block (displacement components of the block centroid and as many parameters needed to describe the strain field).

The global stiffness matrix with $(NB*n)*(NB*n)$ terms, contains $n*n$ submatrices $\mathbf{K}_{ii}$ and $\mathbf{K}_{ij}$ for NB blocks

$$
\begin{bmatrix}
\mathbf{K}_{11} & \mathbf{K}_{12} & \Lambda & \mathbf{K}_{1m} \\
\mathbf{K}_{21} & \mathbf{K}_{22} & \Lambda & \mathbf{K}_{2m} \\
\mathrm{M} & \mathrm{M} & \mathrm{O} & \mathrm{M} \\
\mathbf{K}_{m1} & \mathbf{K}_{m2} & \Lambda & \mathbf{K}_{mm}
\end{bmatrix}
\begin{bmatrix}
\mathbf{D}_{1} \\ \mathbf{D}_{2} \\ \mathrm{M} \\ \mathbf{D}_{m}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{F}_{1} \\ \mathbf{F}_{2} \\ \mathrm{M} \\ \mathbf{F}_{m}
\end{bmatrix}
\qquad (1)
$$

$$
\underline{\underline{\mathbf{K}}} \qquad \underline{\underline{\mathbf{D}}} = \underline{\underline{\mathbf{F}}}
$$

where the nonzero submatrices $\mathbf{K}_{ij}$ are present only if and when the blocks i and j are in *active* contact, and $\mathbf{D}$ comprises deformation variables of all blocks considered in the system.

2 Active Contact Constraints

Depending on the inter-block contact stress state, the contact conditions either

- *allow sliding with no penetration and no tension* or
- *impose a no sliding, no penetration and no tension condition.*

These conditions can be interpreted as block displacement constraints, which is algorithmically reduced to an interaction problem between a vertex of one block, with the side of another

Assuming deformation increments of the two blocks are denoted by $\mathbf{D}_i$ and $\mathbf{D}_j$ respectively, the penetration in the direction normal to the block side can be expressed as a function of these deformation increments by

$$
d = A / L + \mathbf{G}^T \mathbf{D}_i + \mathbf{H}^T \mathbf{D}_j \qquad (2)
$$

Various algorithmic approaches can be identified depending on how the imposition of the no penetration condition (i.e. $d = 0$ for all active contacts) is imposed.

The simplest approach, initially utilised by Shi (1988) adopted the *penalty format,* where the contact spring with a penalty stiffness p is placed between the vertex of one block and the side of the other, implying a contact force equal to $f = pd$. The presence of penalty springs, at all positions of active contact, contributes to the overall strain energy of the system, which in turn affects the system stiffness matrix, as well as the load vector. This formulation leads to a nonlinear iterative scheme

$$\left(\underline{\underline{\mathbf{K}}} + \underline{\underline{\mathbf{K}}}'\left(p, d\left(\underline{\underline{\mathbf{D}}}\right)\right)\right)\underline{\underline{\mathbf{D}}} = \underline{\underline{\mathbf{F}}} + \underline{\underline{\mathbf{F}}}'\left(p, d\left(\underline{\underline{\mathbf{D}}}\right)\right) \tag{3}$$

which proceeds until the global equilibrium is satisfied (norm of the out of balance forces is within some tolerance) while at the same time a *near zero* penetration condition is satisfied at all active contact positions. The convergence process may sometimes be very slow, as both activation and deactivation of contacts during the iteration process is possible. The advantage of the penalty format lies in its simplicity, as the number of system variables does not change and the changes of the secant stiffness matrix are obtained iteratively by augmenting components of the stiffness matrix with components arising from the potential energy of any active contact penalty springs. The convergence of the solution algorithm depends highly on the choice of the penalty term and the process may often lead to ill conditioned matrices, when the very large penalty term is employed to ensure the penetration remains close to zero. However, a non zero penetration is required for a contact force to be present at all. The penalty formulation typically requires an auxiliary calculation of contact forces, as a direct evaluation arising from directly multiplying a near infinite penalty term with a near zero penetration at a converged state is cleary very approximate.

In the *Lagrange multiplier method*, the Lagrange multipliers associated with constraint equations can be interpreted physically as the actual contact forces at all active contacts which are in turn additional unknowns of the problem. The system of equations is augmented by the presence of any active contact and the number of unknowns to be solved for at every increment will clearly be variable as the solution progresses, and contacts are activated and deactivated.

In the case of only contact between blocks i and j being active, only one additional constraint equation exists, i.e.

$$\lambda d = \lambda\left(A / L + \mathbf{G}^T \mathbf{D}_i + \mathbf{H}^T \mathbf{D}_j\right) = 0 \tag{4}$$

which in combination with (1) leads to

$$
\begin{array}{c}
\begin{matrix} \quad & i & \quad & j & \quad & \quad \end{matrix} \\
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{0} \\
\cdot & \mathbf{K}_{ii} & \cdot & \mathbf{K}_{ij} & \cdot & \mathbf{G} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{0} \\
\cdot & \mathbf{K}_{ji} & \cdot & \mathbf{K}_{jj} & \cdot & \mathbf{H} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{0} \\
\cdot & \mathbf{G}^T & \cdot & \mathbf{H}^T & \cdot & 0
\end{bmatrix}
\end{array}
\begin{bmatrix} \cdot \\ \mathbf{D}_i \\ \cdot \\ \mathbf{D}_j \\ \cdot \\ \lambda \end{bmatrix}
=
\begin{bmatrix} \cdot \\ \mathbf{F}_i \\ \cdot \\ \mathbf{F}_j \\ \cdot \\ -A/L \end{bmatrix}
\qquad (5)
$$

$$
\underline{\underline{\mathbf{K}}}_{\lambda} \qquad \underline{\underline{\mathbf{D}}}_{\lambda} = \underline{\underline{\mathbf{F}}}_{\lambda}
$$

The major disadvantage of the method is that *every* additional active contacts leads to the introduction of an additional Lagrange multiplier λ_i resulting in a continuous change in the number of system equations. Upon convergence, the formulation satisfies the contact conditions exactly (or within some small tolerance) , the contact forces are obtained directly as a part of the solution vector and no auxiliary computations are needed to evaluate the contact stress resultants.

The procedure has a major disadvantage as the resulting system matrix contains a zero sub-matrix associated with λ_is and it may not be positive definite, requiring the use of a special matrix pseudo inversion procedure.

The most advanced treatment of contact constraints in the DDA context is the *Augmented Lagrangian Method,* which has been advocated by Lin (1995) and Amadei et al (1995), where an iterative combination of a Lagrangian multiplier and a contact penalty spring is utilised. The ensuing iterative procedure, to obtain the correct contact forces, proceeds until the penetration distance and the norm of the out of balance forces is not smaller than some specified norm.

Consecutive iterates for λ are obtained from

$$
\lambda_{i+1} = \lambda_i + pd\left(\underline{\underline{\mathbf{D}}}_{i+1}\right) \qquad (6)
$$

where the penalty term acts as the solution accelerator parameter.

The contribution to the system potential energy from active contacts (both Lagrange multipliers and penalty springs) leads to changes both in the system stiffness matrix, as well as the right hand side load vector. The associated set of nonlinear equations can be written as

$$\left(\underline{\underline{\mathbf{K}}} + \underline{\underline{\mathbf{K}}}'\left(p, d\left(\underline{\underline{\mathbf{D}}}_i\right)\right)\right)\underline{\underline{\mathbf{D}}}_{i+1} = \\ \underline{\underline{\mathbf{F}}} + \underline{\underline{\mathbf{F}}}'\left(p, d\left(\underline{\underline{\mathbf{D}}}\right)\right) + \underline{\underline{\mathbf{F}}}\left(\lambda_i\right) \tag{7}$$

which requires an update for $\lambda_{i+1} = \lambda_i + pd\left(\underline{\underline{\mathbf{D}}}_{i+1}\right)$ between each iteration. As the value of the last iterate for λ_i is known and appears on the right hand side, the number of global unknowns does not change and the system size to solve for the new iterate $\underline{\underline{\mathbf{D}}}_{i+1}$ remains constant, comprising only the block deformation variables. The components of the current stiffness matrix are affected by the current solution vector $\underline{\underline{\mathbf{D}}}_i$ (which describe the current penetrations $d\left(\underline{\underline{\mathbf{D}}}_i\right)$ at all active contact positions), whereas the load vector comprises a contribution from the previous iterate of contact forces λ_i. As the number of unknowns does not increase, the method retains the advantage present the penalty method. The rate of convergence of the ensuing iterative procedure to obtain the contact forces is clearly controlled by the penalty stiffness term and the iteration process is ended when the norm of the out of balance forces in between the two iterates is less than a specified tolerance limit.

3 Fracturing in Discontionuous Deformation Analysis

The DDA formulation naturally deals with discontinuities along block boundaries, in the sense of a Mohr-Coulomb (with or without the tension cut-off) criterion. The algorithimic argumentation is simple, as the normal and tangential contact springs, which are added to the system whenever a vertex is in the vicinity of a vertex or a side of another block, are released if a pointwise Mohr Coulomb or tension cut off condition (in terms of stress resultants) is violated. It has already been said that the improvements of the DDA formulation use various ways of enhancing the description of the block deformability. Similarly to the combined FEM/DEM formulations, the DDA sub-block framework assumes that each block is subdivided into sub-blocks and the behaviour of each sub-block is again restricted to the constant strain state. The additional constraint conditions in between sub-block boundaries remain valid at all times and their sole purpose is to enhance the description of the block deformability.

Block fracturing algorithm through block centroid has been proposed recently by Lin in 1995, in the context of rock fracture, comprising again the Mohr-Coulomb fracturing criterion with a tension cut-off, where the newly formed discontinuities are introduced and which are further treated in the same way as the original discontinuity planes.

Several discontionuous modelling attempts in modelling fragmentation of concrete structures and concrete protection covers have been reported. Recently Kitoh (1997) conducted a series of block size sensivity tests, exploring both the geometric size effect on fracture for series of normalised plain concrete beams, as well as the influence of the discretisation effect, resulting from different block sizes. Similar Voronoi tessalation has been utilised to illustrate the potential of the DDA based fracturing models.

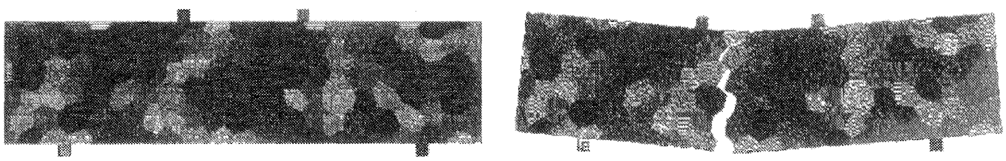

Fig 2. Kitoh Plain Concrete Beam (h=100mm),
DDA coarser discretisation (130 blocks),
initial geometry and failure mode

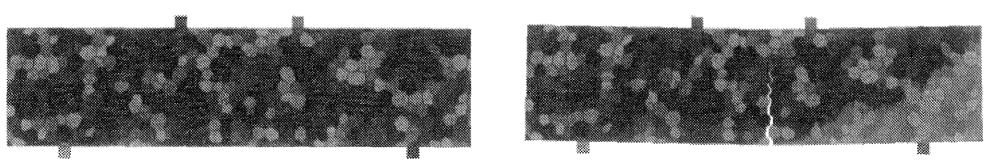

Fig 3. Kitoh Plain Concrete Beam (h=100mm),
DDA finer discretisation (455 blocks),
initial geometry and failure mode

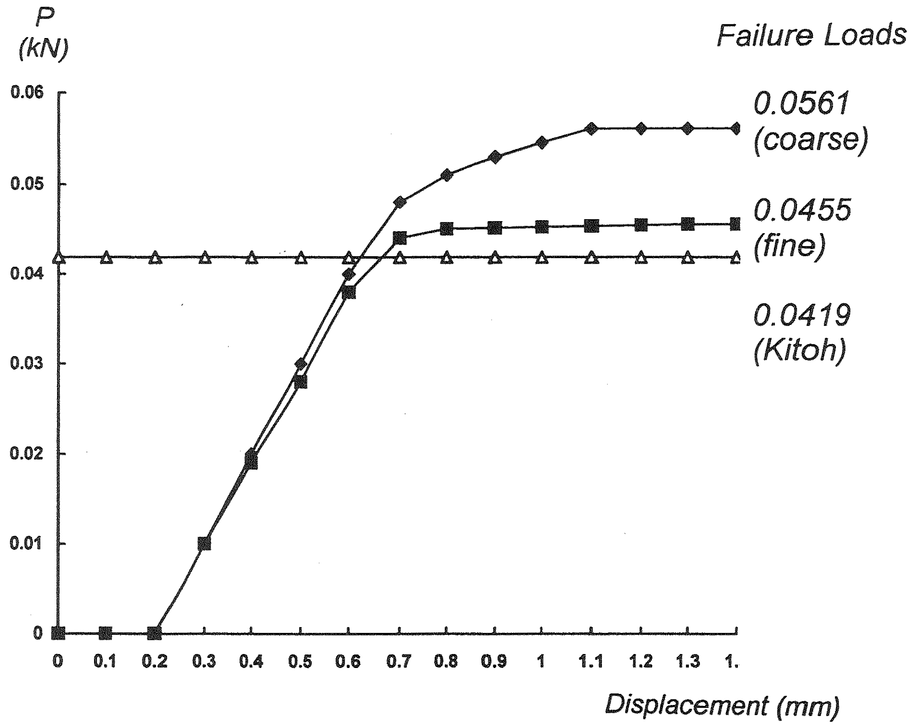

Fig 4. Kitoh Plain Concrete Beam
Comparison of P-δ diagrams for the coarser and finer DDA discretisation, together with RSBM failure load

The Kitoh plain concrete beam four point bending problem (*h=100mm*) is here modelled with two DDA discretisations. The failure load predictions from a course discretisation using 130 simply deformable blocks is compared to the failure load prediction from a model comprising 455 simply deformable blocks and both results are set against the failure load prediction reported by Kitoh. The block material characteristics and the interface material law are identical in both cases

$E = 27.5$ MPa
$\nu=0.20$
$c=0.0047$ MPa
$f_t=0.0029$ MPa
$\varphi=37°$

which values correspond to the data adopted by Kitoh in his RSBM analysis. The analysis has been conducted as force controlled, with an adaptive load incrementation in approaching failure.

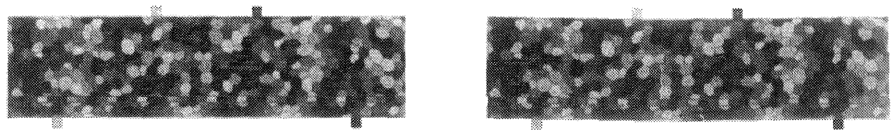

Fig 5. Hypothetical Reinforced Concrete Beam
same geometry as Kitoh Plain Concrete Beam (h=100mm),
DDA finer discretisation, 455 blocks and failure mode

The DDA analysis of a hypothetical reinforced concrete beam (Fig 5) comprises a steel/concrete interface law of c=0.00188 MPa, f_t=0.00145 MPa, φ=22.5°, whereas the steel/steel interface is fixed, and the concrete/concrete interface is identical to the plain concrete beam analysis. Final failure mode indicates a shift of an initial tensile crack, due to the bond slip failure on the concrete/steel interface.

Further example (Fig 6) illustrates the DDA discretisation (704 simply deformable blocks) of the RILEM TC90-FMA pull-out benchmark problem and the final mode of failure, for the case of k=∞, using the same data.

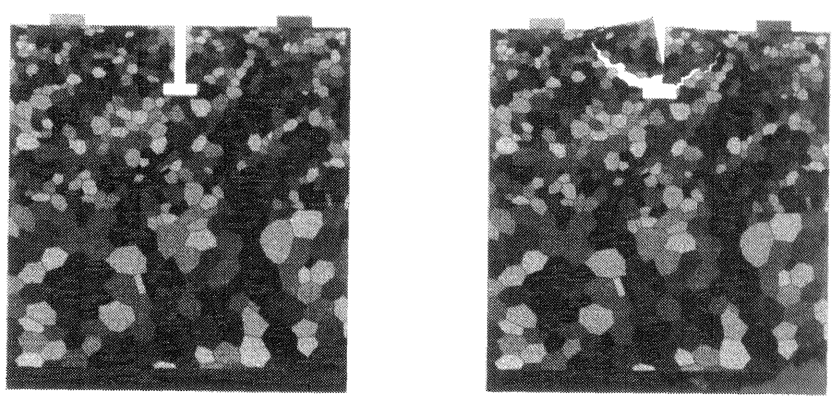

Fig 6. RILEM Pull Out Test
DDA discretisation (704 simply deformable blocks) and failure mode
(material data same as Kitoh Beam)

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