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Nonlocal Weibull theory and size effect in failures at fracture initiation

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ABSTRACT: If the structure fails at fracture initiation from a smooth surface, and if the geometry is positive and the structure size tends to infinity, the Weibull-type size effect must be exactly followed, even if the material is quasibrittle. A nonlocal generalization of Weibull theory that satisfies this condition is presented. It predicts the probability of failure of unnotched structures that reach the maximum load before a large crack forms, as is typical of the test of modulus of rupture. The probability of failure at a material point is assumed to be a power function of the average strain in the neighborhood of that point. The recent nonlocal generalization of Weibull theory is reviewed and shown to exhibit the correct large-size properties. The size effect on the modulus of rupture (bending strength) and its statistics are analyzed and comparisons with extensive test data are presented.

1 INTRODUCTION

Prior to the 1990's, it was commonplace in civil engineering design to assume the maximum load of structures to be governed by the strength of the material. Only sometimes the possibility of a purely statistical, classical size effect of Weibull (1939) was admitted. But no attention was paid to the possibility of a deterministic size effect. As became clear in the early 1980's, the size effect on the nominal strength of quasibrittle structures is in most instances predominantly deterministic. It is caused by stress redistributions and energy release associated with either the growth of a large fracture process zone (FPZ) or a long stable crack (Bažant 1984, Bažant & Chen 1997, Bažant & Planas 1998). More than two decades ago, the finite element calculations with the cohesive (or fictitious) crack model by Hillerborg et al. (1976) revealed the presence of a strong deterministic size effect engendered by stress redistribution within the cross section due to softening inelastic response of the material in a boundary layer of cracking near the tensile face. At the same time, the necessity of a deterministic size effect was indicated by energy analysis of stability of softening damage against localization and spontaneous propagation (Bažant 1976, 1984). A detailed finite element analysis of the deterministic size effect on the modulus of rupture was carried out, with the cohesive crack model, by Petersson (1981). He numerically demonstrated that the deterministic size effect curve terminates with a horizontal asymptote. But he also observed that, for very deep beams, for which the deterministic size effect asymptotically disappears, the classical Weibull-type statistical size effect must take over.

It has been argued that a sound probabilistic theory of quasibrittle failure must asymptotically approach the Weibull theory with the weakest link model (extreme value statistics) in the case that the ratio of structure size D to the characteristic length l of the material tends to ∞. The stochastic finite element method (SFEM), in which the role of l is played by the autocorrelation length of the random field of material strength, does not satisfy this basic requirement, while the proposed theory does.

Since concrete is a highly random material, the statistical size effect must of course get manifested in some way. A recent study of Bažant & Novák (2000a,b) resulted in a statistical structural analysis model that takes into account the post-peak strain softening of the material and calculates the failure probability from the redistributed stress field using the nonlocal Weibull approach of Bažant & Xi (1991). A simple formula for the size effect on modulus of rupture incorporating both the deterministic energetic and the statistical causes has been presented by Bažant & Novák (2000c). This paper presents the main results of these studies.

2 NONLOCAL WEIBULL THEORY

The Weibull integral for probability P_f of structural failure (Bažant & Planas 1998, ch. 12) was reformulated by Bažant & Novák (2000a,b) in a nonlocal form. In this reformulation, the local stresses are replaced by the nonlocal (spatially averaged) strains multiplied by the modulus of elasticity, as proposed by Bažant & Xi (1991). Then the multi-dimensional generalization of Weibull integral may be written as:

$$P_f = 1 - \exp\left\{-\int_V \sum_{i=1}^{n} \left\langle \frac{\overline{\sigma}_i(\boldsymbol{x})}{\sigma_0} \right\rangle^m \frac{dV(\boldsymbol{x})}{V_r}\right\} \quad (1)$$

where n = number of dimensions (1, 2 or 3), σ_0 = Weibull scaling parameter, m = Weibull modulus, V_r = representative volume of material (having the dimension of material length), σ_i = principal stresses ($i = 1, ...n$), and an overbar denotes nonlocal averaging. The failure probability now depends no longer on the local stresses $\sigma_i(\boldsymbol{x})$ but on the nonlocal stresses $\overline{\sigma}_i(\boldsymbol{x})$ which are the results of some form of spatial averaging of strains; for details see Bažant & Xi (1991), Bažant & Planas (1998, ch. 12), and Bažant & Novák (2000a,b). In the case of an unreinforced simply supported symmetric beam with a symmetric uniaxial stress field treated as two-dimensional, (1) becomes:

$$P_f = 1 - \exp\left\{-\frac{2}{V_r}\int_0^{\frac{L}{2}} \int_{-a}^{\frac{D}{2}} \left[\frac{\overline{\sigma}(x,y)}{\sigma_0}\right]^m dx dy\right\} \quad (2)$$

where L = span of the beam, D = size (height) of the beam and a = shift of the neutral axis of beam caused by distributed cracking.

It might seem that the analysis of strain-softening would call for using finite elements. In the present problem of beam bending, however, this is unnecessary because only the states before a crack forms are of interest. The softening zone, restrained by the adjacent material that is in an elastic state, does not yet localize, remaining distributed over a long portion of the beam. Therefore, the classical hypothesis of cross sections remaining planar is a good approximation.

3 ENERGETIC-STATISTICAL FORMULA FOR MODULUS OF RUPTURE

The size effect on the modulus of rupture has been shown first to follow the energetic formula (Bažant & Novák 2000c):

$$f_r = f_{r\infty}\left(1 + \frac{rD_b}{D}\right)^{1/r} \quad (3)$$

in which D_b has the meaning of the thickness of the boundary layer of cracking;

$$D_b = \left\langle \frac{-c_f g''(0)}{4g'(0)} \right\rangle \qquad (D_b > 0) \quad (4)$$

In the last expression, the signs $\langle .. \rangle$, denoting the positive part of the argument, have been inserted [$\langle X \rangle$ = Max$(X, 0)$]. The reason is that $g''(0)/g'(0)$ can sometimes be positive, in which case there is no size effect, and this is automatically achieved by setting $D_b = 0$. In the modulus of rupture test, $g''(0)/g'(0) < 0$ and $D_b > 0$. Note that for uniform tension (zero stress gradient, as in the direct tensile test) there is no deterministic size effect according to (4) because $g''(0) = 0$ or $D_b = 0$.

For $r = 2$,

$$f_r = \sigma_N = \sqrt{A_1 + \frac{A_2}{D}} \quad (5)$$

in which

$$A_1 = f_{r\infty}^2 = \frac{EG_f}{[c_f g'(0)]^2}, \quad (6)$$

$$A_2 = f_{r\infty}^2 q_1 = 2 f_{r\infty} D_b = -\frac{EG_f g''(0)}{2c_f [g'(0)]^3} \quad (7)$$

Formula (5) was proposed and used to describe some size effect data by Carpinteri et al. (1994, 1995). These authors named this formula the 'multifractal' scaling law (MFSL) and tried to justify it by fracture fractality using, however, strictly geometric (non-mechanical) arguments. This name, though, seems questionable because, as shown in Bažant (1997 and 1998), the mechanical analysis of fractality leads to a formula different from (5) (this is the case whether one considers the invasive fractality of the crack surface or the lacunar fractality of microcrack distribution in the fracture process zone). No logical mechanical argument for the size effect on σ_N to be a consequence of the fractality of fracture has yet been offered.

To achieve greater flexibility in the modeling of test data for small sizes, the deterministic energetic formula may be further generalized as:

$$f_r = f_{r\infty}\left(\frac{D + r(s+1)D_b}{D + rsD_b}\right)^{1/r} \quad (8)$$

where s is a non-negative constant.

The energetic formula has been modified based on the fact that the large size asymptote must approach Weibull-type size effect. The following energetic-statistical formula (Bažant & Novák 2000c) has been proposed:

$$f_r = f_{r\infty}\left[\left(\frac{D_b}{D}\right)^{rn/m} + \frac{rD_b}{D}\right]^{1/r} \quad (9)$$

where $f_{r\infty}$, D_b, r and m are positive constants, representing unknown empirical parameters, and n is the number of dimensions in geometric similarity. The data fitting with the new formula (9) reveals that, for concrete and mortar, the Weibull modulus $m \approx 24$ rather than 12, the value widely accepted so far (Bažant & Novák 2000c). This means that, for extreme sizes, the nominal strength (modulus of rupture) decreases, for two-dimensional (2D) similarity ($n = 2$), as the $-1/12$ power of the structure size (in contrast to the $-1/6$ power that has generally been assumed so far).

4 NUMERICAL EXAMPLES: COMPARISON WITH EXISTING TEST DATA

The present theory has been compared with the most important data sets found in the literature (Bažant & Novák 2000a,b,c). The details on the extensive test data used and on the comparative calculations can be found in the referenced articles. Here only selected comparisons are included, among many results.

4.1 *Estimation of cumulative probability distribution function*

The Weibull-type integral makes it possible to estimate the failure probabilities corresponding to different load levels. Covering the full range of probabilities, one can estimate the probability distribution function for the modulus of rupture. The proper load levels are such that the entire range of the cumulative probability distribution function from 0 to 1 could be covered almost regularly. Thus it is efficient to use the idea of the stratified sampling called Latin hypercube sampling (McKay et al. 1979, Novák et al. 1998).

The probability distribution functions of the ratio of the modulus of rupture to the strength are plotted in Figure 1 for different sizes. The sample size $N = 16$ has been chosen for calculations — 16 different probabilities, which are taken as the input into the nonlocal Weibull model. As expected, the steepness increases with an increasing size, which means that the scatter decreases with the size. This agrees with the well-known fact that the statistical correlation of strength imposed by averaging has a major influence only for small sizes. Such trends for the distribution functions were already in general sketched by Shinozuka (1972).

An important source of statistical information are Koide et al.'s (1998) tests of 279 plain concrete beams in four-point bending, aimed at determining the influence of the beam length L on the flexural strength of beams. Koide's are excellent data which allow comparing the cumulative probability distribution function (CPDF) of the maximum bending moment M_{max} at failure, over its full range. The data points in Figure 2 show the empirical cumulative probability density functions for one selected span (Koide's series C). A good agreement with Koide et al.'s data has been achieved. The calculations indicate a decrease of the flexural strength as the span increases. Notice the similar trends in Figure 2 (limited sizes), and more generally in Figure 1 (an extremely broad range of sizes).

4.2 *Fitting of energetic-statistical formula with test data*

Fitting of the formula to the main test data sets available in the literature showed an excellent agreement, with a rather small coefficient of variation of errors of the formula compared to the test data. The result is shown in Figure 3. The corresponding coefficient of variation is $\omega = 0.023$ and the optimum values of the parameters are $f_{r\infty} = 3.68$ MPa, $D_b = 15.53$ mm and $r = 1.14$.

Furthermore, the new formula was verified numerically by the nonlocal Weibull theory. The result of nonlinear fitting of formula (9) using the nonlocal solutions of failure probability (medians of modulus of rupture) of the beam is presented in Figure 4. The corresponding parameters are $f_{r\infty} = 3.76$ MPa, $D_b = 48.66$ mm and $r = 1.28$. As it can be seen, both curves are very close. This favorable comparison supports (but of course does not prove) the correctness of the present energetic-statistical size effect formula (9), as well as the nonlocal Weibull material model.

4.3 *Fitting of generalized deterministic energetic formula: small sizes aspect*

The same strategy and the same experimental data have been used to fit formula (8). A straightforward fitting of additional parameter s would lead to a fitting of 4 unknowns parameters, which will be very unstable (ill-conditioned) and almost impossible. Therefore a different strategy has been accepted: The value

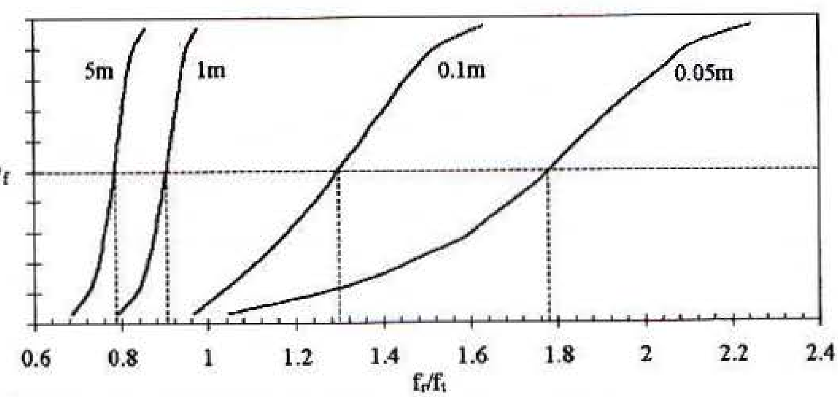

Figure 1. Cumulative probability distributions of modulus of rupture for different sizes.

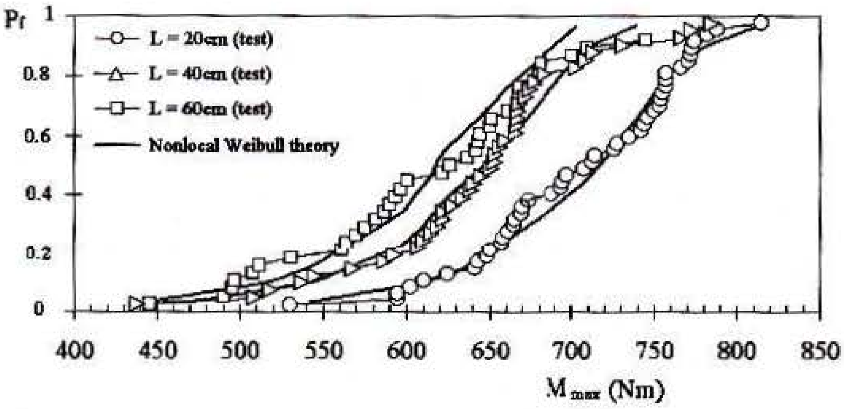

Figure 2. Comparison of CPDF of maximum bending moment from Koide's (1998), series C, of four-point bending tests and from the probabilistic nonlocal theory.

of s is forced to be constant during the iterative fitting approach. Various fixed values s are considered and the coefficient of variation ω of errors is calculated. Then the most suitable value of s is the value that minimizes ω.

The result is that the coefficient of variation of errors increases as parameter s increases, as is obvious from Figure 5. So it seems that most suitable value of s is zero. But this conclusion might be distorted because there are only very few test data in the range of very small sizes.

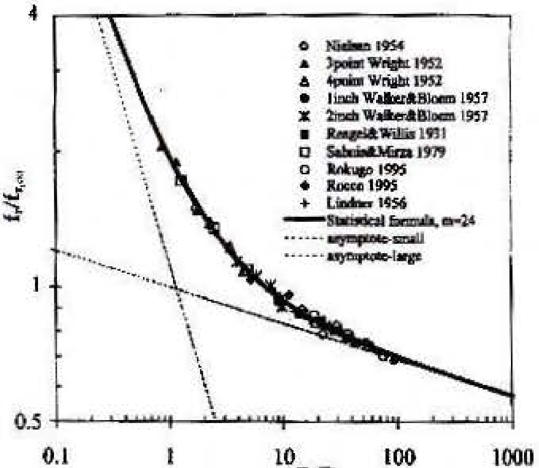

Figure 3. Optimum fit to existing test data.

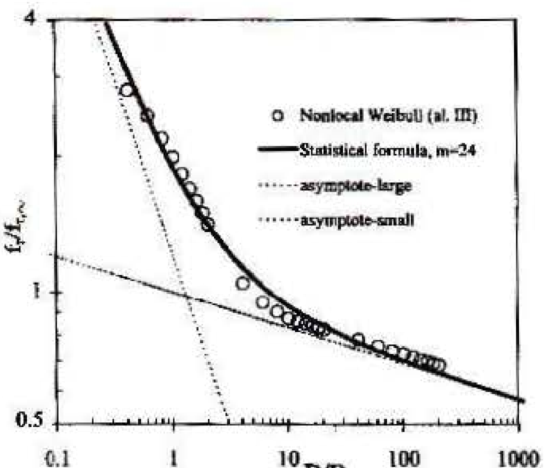

Figure 4. Optimum fit to the nonlocal Weibull theory.

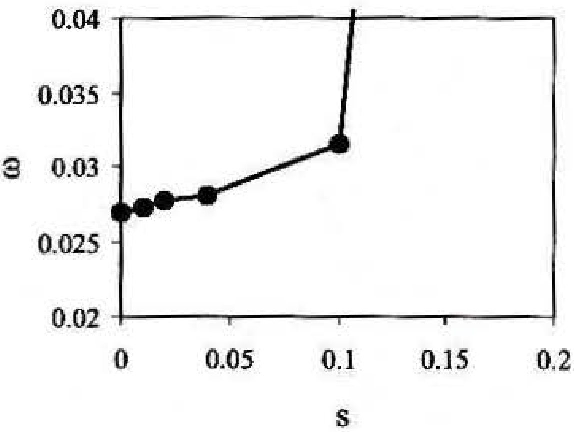

Figure 5. Change of coefficient of variation of errors with parameter s.

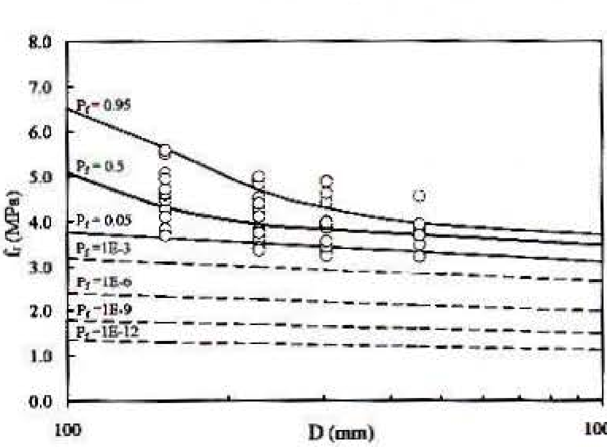

Figure 6. Failure probabilities vs. size for data of Lindner & Sprague (1956).

4.4 *Small failure probabilities*

One advantage of the present approach is that small failure probabilities can be estimated without an increase of computational time (as is typical for Monte-Carlo based approaches in classical reliability engineering). The same approach (Weibull integral) is used for estimation of the median ($P_f = 0.5$) and e.g. for $P_f = 10^{-9}$. A broad range of failure probabilities is shown, as an illustration, in Figure 6, for data of Lindner & Sprague (1956). Naturally, for small failure probabilities, the curves approach the Weibull type of size effect.

5 CONCLUSIONS

1. In the nonlocal generalization of Weibull theory the failure probability of a small material element is a function of the nonlocal (spatially averaged) continuum variables rather than the local stress. This generalization can be applied to unnotched specimens, and in particular to the test of the modulus of rupture (flexural strength).

2. A new generalized formula (9) that amalgamates the energetic and statistical size effects for failures at crack initiation has been developed. Its correctness is supported by good agreement with structural analysis according to the statistical nonlocal material model.

3. The present models agree well with the test data sets found in the literature. The benefit of the present theory is the possibility to predict the full probability distribution of structural strength, and in particular the modulus of rupture.

4. According to the best writers' knowledge, the Weibull-type size effect has not yet been reproduced by SFEM. [1]

5. Compared to the existing stochastic finite element approaches, a great simplification is achieved by the fact that the nonlocal structural analysis with strain softening can be conducted deterministically because the probabilistic analysis is separated from the stress analysis, similar to the classical Weibull theory.

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REFERENCES

Bažant, Z.P. 1976. Instability, ductility, and size effect in strain-softening concrete. *J. Engrg. Mech. Div.*, Am. Soc. Civil Engrs., 102, EM2: 331–344 (disc. 103, 357–358, 775–777, 104, 501–502).

Bažant, Z.P. 1984. Size effect in blunt fracture: Concrete, rock, metal. *J. of Engrg. Mechanics*, ASCE, 110: 518–535.

Bažant, Z.P. 1997. Scaling of quasibrittle fracture: Hypotheses of invasive and lacunar fractality, their critique and Weibull connection. *Int. J. of Fracture*, 83(1): 41–65.

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[1]The decisive parameter in SFEM is the correlation length which prescribes spatial correlation over the structure. The correlation length modifies the size effect curve in the region where this parameter is smaller than the element size. There is a clear relationship—the larger the correlation length, the stronger the spatial correlation of strength along the structure, and consequently the smaller the decrease of nominal strength of the structure with its increasing size. Problems occur in trying to obtain the extreme value asymptote using the random field approach. Approximately, the requirement is that the ratio of the correlation length to the element size should not drop bellow one. This poses a major obstacle to using SFEM for describing the size effect, especially for large structure sizes. Some advances in this topic were achieved by several authors, e.g. Carmeliet & Hens (1994). But these authors usually confine their studies to the region of reasonable sizes. The ratio of the correlation length to the element size implies some limitations. To obtain the extreme value asymptote using the random field approach, the number of discretization points (e.g. nodes in finite element method) should increase proportionally to the increase of structure size. In other words, keeping the same element size for different sizes of the structure is preferable to the alternative of keeping the same number of elements. This requirement for size effect studies using SFEM can be crucial or even impossible to adopt: The number of elements can become extremely large! In the nonlocal Weibull theory there is no such limitation: For any mesh $dV(x)$ ($dx\,dy$ for 2D problems), the Weibull integral is calculated through an algebraic sum, and there is always a correct increase of the failure probability with an increasing structure size for a certain load (which leads to a size effect of Weibull type for very large sizes).

Bažant, Z.P. 1998. Statistical and fractal aspects of size effect in quasibrittle structures: Conspectus of recent results. *Proc., ICOSSAR-97, Kyoto*: 1255–1262. Rotterdam: Balkema.

Bažant, Z.P. & Chen, E.-P. 1997. Scaling of structural failure. *Applied Mechanics Reviews ASME* 50(10): 593–627.

Bažant, Z.P. & Novák, D. 2000a,b. Probabilistic nonlocal theory for quasi-brittle fracture initiation and size effect. I. Theory and II. Application. *J. of Engrg. Mechanics ASCE* 126(2): 166–174, 175–185.

Bažant, Z.P. & Novák, D. 2000c. Energetic-statistical size effect in quasi-brittle failure at crack initiation. *ACI Materials Journal*, 97(3): 381–392.

Bažant, Z.P. & Planas, J. 1998. *Fracture and size effect in concrete and other quasibrittle materials*, CRC Press, Boca Raton, Florida, and London, U.K. (chapter 12).

Bažant, Z.P. & Xi, Y 1991. Statistical size effect in quasibrittle structures: II. Nonlocal theory. *ASCE J. of Engrg. Mech.*, 117(11): 2623–2640.

Carmeliet, J. & Hens, H. 1994. Probabilistic nonlocal damage model for continua with random field properties. *ASCE J. of Engrg. Mech.*, 120(10): 2013–2027.

Carpinteri, A., Chiaia, B. & Ferro, G. 1994. Multifractal scaling law for the nominal strength variation of concrete structures, in *Size effect in concrete structures* (Proc., Japan Concrete Institute International Workshop, held in Sendai, Japan, 1993), ed. by M. Mihashi, H. Okamura and Z.P. Bažant, E & FN Spon, London-New York: 193–206.

Carpinteri, A., Chiaia, B. & Ferro, G. 1995. Size effects on nominal tensile strength of concrete structures: multifractality of material ligaments and dimensional transition from order to disorder. *Materials and Structures* 28(7): 311-317.

Hillerborg, A., Modéer, M. & Petersson, P.-E. 1976. Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cement and Concrete Res.*, 6(6): 773–782.

Koide, H., Akita, H. & Tomon, M. 1998. Size effect on flexural resistance due to bending span of concrete beams. *Fracture Mechanics of Concrete Structures* (Proc., 3rd Int. Conf., FraMCoS-3, held in Gifu, Japan), H. Mihashi and K. Rokugo, eds., Aedificatio Publishers, Freiburg, Germany: 2121–2130.

Lindner, C.P. & Sprague, I.C. 1956. Effect of depth of beams upon the modulus of rupture of plain concrete . *ASTM Proc.*, 55: 1062-1083.

McKay, M.D., Conover, W.J. & Beckman, R.J. 1979. A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics*, 21: 239–245.

Novák, D., Teplý, B. & Keršner, Z. 1998. The role of Latin Hypercube Sampling method in reliability engineering. *Proc., ICOSSAR-97, Kyoto*: 403–409. Rotterdam: Balkema.

Petersson, P.-E. 1981. *Crack Growth and Development of Fracture Zone in Plain Concrete and Similar Materials*. Report No. TVBM-1006, Division of Building Materials, Lund Institute of Technology, Lund, Sweden.

Shinozuka, M. 1972. Probabilistic modeling of concrete structures. *ASCE J. of Engrg. Mech. Division*, 98 (EM6): 1433–1451.

Weibull W. 1939. The phenomenon of rupture in solids. *Proc., Royal Swedish Institute of Engineering Research* (Ingenioersvetenskaps Akad. Handl.) 153, Stockholm: 1–55.