Conservation laws for multiphase fracturing materials

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ABSTRACT: A defect is simulated as an abrupt change of elastic properties in the material and the conservation laws, discovered by Knowles and Sternberg, are deduced and extended on the basis of a general formula for the dissipation, associated with the propagation of elastic phase transition fronts, recently contributed. Moreover, by removing the assumptions of a homogeneous elastic medium, under homogeneous anelastic deformation and no body forces, further invariance conditions, suitable for the evaluation of the dissipation associated with an evolving defect, are provided.

1 INTRODUCTION

Defect propagation in continuous bodies is a topic of major interest in fracture mechanics both from the theoretical, the applied and the computational point of view. According to Eshelby (Eshelby, 1975), a defect is an *elastic inhomogeneity* while, according to Maugin (Maugin, 1993), a defect *creates a singularity in the elastic field*. In evaluating the elastic release rate induced by defect's propagation in a material, a defect can be effectively simulated as an abroupt change of elastic properties. Its evolution is thus described by the motion of a shock wave front on which the elastic phase transition occurs (Romano et al., 2005). Indeed, across the front the displacement field is continuous while its derivative may undergo a discontinuity jump. When the propagation of a defect is described by a rigid body motion of a shock wave front, the characteristic vectors associated with the elastic release rate may be defined by duality. Several treatments based on Eshelby's energy-momentum tensor, and concerning the evaluation of the force acting on defects traveling in an elastic medium, are reported in the literature. Eshelby's formula (Eshelby, 1951), (Eshelby, 1970), (Eshelby, 1975) for the evaluation of the force acting on a defect was based on subsequent cut and fitting after a rigid translation of a surface embodying the defects. Maugin (Maugin, 1993) has provided a proof of this formula, making reference to Eshelby's treatment. Here the general formula contributed in (Romano et al., 2005) is applied to the evaluation of the dissipation associated with the evolution of translating, rotating and homothetically transforming defects in a multi-phase material body. In this context, some conservation laws (Knowles and Sternberg, 1972) are discussed and subsequently extended. These laws are then specialized by performing a geometrically linearized analysis in agreement with (Knowles and Sternberg, 1972), (Green, 1973), (Budiansky and Rice, 1973). In Appendix it is shown that, in an isotropic elastic medium, the Eshelby's energy-momentum tensor is symmetric. This property plays a foundamental role in the evaluation of the dissipation associated with a rotating defect.

2 DISSIPATION FORMULA DUE TO ELASTIC PHASE-TRANSITION PHENOMENA

Let $\{\mathbb{S}, \mathbf{g}\}$ be the euclidean space with the standard inner product. Let us denote by $\mathbb{M} \subset \mathbb{S}$ and $\boldsymbol{\varphi}(\mathbb{M})$, with $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{S})$, two placements of a body modeled as a 3-D CAUCHY's continuum. To get a general picture of an elastic multi-phase material body, we consider a partition of the reference configuration into a finite family $\mathcal{T}(\mathbb{M})$ of open non-overlapping domains, such that the union of their closures covers $\mathbb{M}$. Each element $\mathcal{P}$ of the partition $\mathcal{T}(\mathbb{M})$ is made of a single-phase material whose elastic properties are described by a free energy density:

$$W_{\mathbf{m}}(\mathbf{D}(\boldsymbol{\varphi})_{\mathbf{m}}, \boldsymbol{\Delta}_{\mathbf{m}}) \quad \forall\, \mathbf{m} \in \mathcal{P}\,.$$

Denoting by $d\boldsymbol{\varphi}(\mathbf{m})$ the differential of the configuration map at $\mathbf{m} \in \mathbb{M}$, $\mathbf{D}(\boldsymbol{\varphi})_{\mathbf{m}} := (d\boldsymbol{\varphi}^T d\boldsymbol{\varphi})_{\mathbf{m}}$ is the Piola-Green operator and $\boldsymbol{\Delta}_{\mathbf{m}}$ is a symmetric operator which describes the anelastic deformation. Elastic phase-transition phenomena, modeled as abrupt changes of the elastic properties of the material across discontinuity surfaces (shock waves) propagating in the body, are described by a flow $\boldsymbol{\chi}_{\tau,t} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$

which modifies the reference partition $\mathcal{T}(\mathbb{M})$ at time t into an evolving one $\mathcal{T}_\tau(\mathbb{M}) := \boldsymbol{\chi}_{\tau,t}(\mathcal{T}_t(\mathbb{M}))$ at time $\tau \in I$. Dropping the dependence on the point $\mathbf{m}$, we write the free energy density as:

$$W_\tau := W(\mathbf{D}(\boldsymbol{\varphi})_\tau, \boldsymbol{\Delta}_\tau) \qquad \tau \in I\,,$$

in which $\mathbf{D}(\boldsymbol{\varphi})_\tau$ and $\boldsymbol{\Delta}_\tau$ are the Piola-Green operator and the anelastic deformation at the time τ. At each point $\mathbf{m}$ of the elements $\mathcal{P}_\tau$ of the partition $\mathcal{T}_\tau(\mathbb{M})$, the time-derivative $\dot{W} := \partial_{\tau=t} W_\tau$ can be evaluated as:

$$\begin{aligned}\dot{W} &= \langle d_1 W, \partial_{\tau=t} \mathbf{D}(\boldsymbol{\varphi})_\tau \rangle_{\mathbf{g}} + \langle d_2 W, \partial_{\tau=t} \boldsymbol{\Delta}_\tau \rangle_{\mathbf{g}} \\ &= \langle d_1 W, \dot{\mathbf{D}}(\boldsymbol{\varphi}) \rangle_{\mathbf{g}} + \langle d_2 W, \dot{\boldsymbol{\Delta}} \rangle_{\mathbf{g}}\,,\end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ is the inner product between linear operators induced by the metric $\mathbf{g}$ and $d_i W \ \ , i = 1\,,2$ are the partial gradients. By imposing the constitutive requirement: $\mathbf{S} := d_1 W$ (symmetric Piola-Kirchhoff stress) and $\mathbf{S_a} := -d_2 W$ (symmetric anelastic stress) we get:

$$\dot{W} = \langle \mathbf{S}, \dot{\mathbf{D}}(\boldsymbol{\varphi}) \rangle_{\mathbf{g}} - \langle \mathbf{S_a}, \dot{\boldsymbol{\Delta}} \rangle_{\mathbf{g}}\,.$$

In the sequel we denote by $\mathcal{I}(\mathbb{M})$ the set of phase-transition interfaces in which the interface between the element $\mathcal{P}^-$ and $\mathcal{P}^+$ is assumed oriented as boundary of the $\mathcal{P}^-$ element and by $[[W]] := [[W_t]] = W^+ - W^-$ the jump across $\mathcal{I}(\mathbb{M})$ (Fig. 1). The dissipation associated with the motion of the

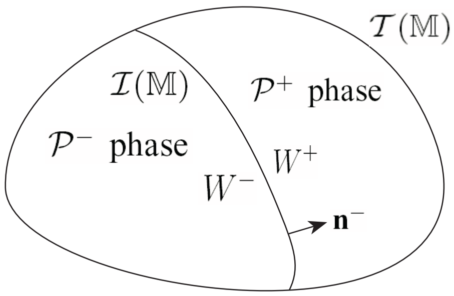

Figure 1. Phase transition interface.

shock wave and with the anelastic deformation can be evaluated on the basis of Maxwell's and Hadamard's kinematic conditions and is given by (Romano et al., 2005):

$$\begin{aligned}\mathcal{D} &= \int_{\mathcal{I}(\mathbb{M})} \mathbf{g}([[W]]\,\mathbf{n}\,, \dot{\boldsymbol{\chi}})\,\boldsymbol{\mu}\mathbf{n} \\ &\quad - \int_{\mathcal{I}(\mathbb{M})} \mathbf{g}(\mathbf{Pn}\,, [[d\boldsymbol{\varphi}]]\,\dot{\boldsymbol{\chi}})\,\boldsymbol{\mu}\mathbf{n} + \int_{\mathcal{T}(\mathbb{M})} \langle \mathbf{S_a}, \dot{\boldsymbol{\Delta}} \rangle_{\mathbf{g}}\,\boldsymbol{\mu} \\ &= \int_{\mathcal{I}(\mathbb{M})} \mathbf{g}([[\mathbf{Y}]]\,\mathbf{n}\,, \dot{\boldsymbol{\chi}})\,\boldsymbol{\mu}\mathbf{n} + \int_{\mathcal{T}(\mathbb{M})} \langle \mathbf{S_a}, \dot{\boldsymbol{\Delta}} \rangle_{\mathbf{g}}\,\boldsymbol{\mu}\,,\end{aligned}$$

where $\mathbf{P} := d\boldsymbol{\varphi}\,\mathbf{S}$ is the Piola operator, $\mathbf{Y} := W\mathbf{I} - d\boldsymbol{\varphi}^T\mathbf{P}$ is Eshelby's energy-momentum tensor, $\dot{\boldsymbol{\chi}} := \partial_{\tau=t}\,\boldsymbol{\chi}_{\tau,t}$ and $\boldsymbol{\mu}\mathbf{n}$ is the area form resulting from the contraction of the volume form $\boldsymbol{\mu}$ and the normal $\mathbf{n} = \mathbf{n}^-$. The vanishing of the divergence of the Eshelby's energy-momentum tensor in each phase of the multi-phase material is at the basis of the invariance properties that will be discussed in the next sections. The general expression of the divergence is given by (Romano et al., 2005):

$$\begin{aligned}\mathbf{g}(\operatorname{div}\mathbf{Y}, \mathbf{h}) = \mathbf{g}(d_3 W, \mathbf{h}) \\ - \langle \mathbf{S_a}, d_{\mathbf{h}}\boldsymbol{\Delta} \rangle_{\mathbf{g}} + \mathbf{g}(\mathbf{b}, d_{\mathbf{h}}\boldsymbol{\varphi})\,, \\ \forall \mathbf{h} \in \mathbb{T}_{\mathbf{m}}\mathbb{M}\,.\end{aligned}$$

Accordingly, in a homogeneous elastic phase, under homogeneous anelastic deformation and no body forces we have that $\operatorname{div}\mathbf{Y} = 0$.

3 DEFECTS

The expressions of the dissipation associated with translating, rotating and homothetically transforming defects in an elastic medium can be obtained by specializing the general formula provided in section 2. To simplify the exposition we will assume that $\dot{\boldsymbol{\Delta}} = 0$.

3.1 Translating defect

Let us consider a defect $\mathcal{Z} \subset \mathbb{M}$ propagating in an elastic medium $\mathbb{M}$ with a translational velocity field $\dot{\boldsymbol{\chi}}(\mathbf{x}) = \dot{\chi}\,\mathbf{d}$, with $\mathbf{d}$ unit vector. The associated dissipation is given by:

$$\mathcal{D} = \int_{\partial\mathcal{Z}} \mathbf{g}([[\mathbf{Y}]]\mathbf{n}, \dot{\boldsymbol{\chi}})\ \boldsymbol{\mu}\mathbf{n} = \dot{\chi} \int_{\partial\mathcal{Z}} \mathbf{g}([[\mathbf{Y}]]\mathbf{n}, \mathbf{d})\ \boldsymbol{\mu}\mathbf{n}\,.$$

Writing the dissipation as $F\,\dot{\chi}$, the driving force F is defined by:

$$\begin{aligned}F &= \int_{\partial\mathcal{Z}} \mathbf{g}([[\mathbf{Y}]]\mathbf{n}, \mathbf{d})\ \boldsymbol{\mu}\mathbf{n} \\ &= \int_{\partial\mathcal{Z}} \mathbf{g}([[W]]\mathbf{n}, \mathbf{d})\ \boldsymbol{\mu}\mathbf{n} - \int_{\partial\mathcal{Z}} \mathbf{g}(\mathbf{Tn}, [[d_{\mathbf{d}}\mathbf{u}]])\ \boldsymbol{\mu}\mathbf{n}\,.\end{aligned}$$

3.2 Rotating defect

Let us now consider the case when the velocity field is rotational: $\dot{\boldsymbol{\chi}}(\mathbf{x}) = \boldsymbol{\omega} \times \mathbf{x} \quad \forall \mathbf{x} \in \mathcal{Z}$ with angular speed $\boldsymbol{\omega} = \omega\,\mathbf{k}$, where $\mathbf{k}$ is an unit vector. The symbol $\times$ denotes the cross product operation defined by $\mathbf{g}(\boldsymbol{\omega} \times \mathbf{x}) := \boldsymbol{\mu}\,\boldsymbol{\omega}\,\mathbf{x}$, in which $\boldsymbol{\mu}\,\boldsymbol{\omega}\,\mathbf{x}$ is the one-form resulting from the contraction of the volume three-form $\boldsymbol{\mu}$ and the vectors $\boldsymbol{\omega}$ and $\mathbf{x}$. The dissipation is then given by:

$$\begin{aligned}\mathcal{D} &= \int_{\partial\mathcal{Z}} \mathbf{g}([[\mathbf{Y}]]\mathbf{n}, \dot{\boldsymbol{\chi}})\ \boldsymbol{\mu}\mathbf{n} = \int_{\partial\mathcal{Z}} \mathbf{g}([[\mathbf{Y}]]\mathbf{n}, \omega\,\mathbf{k} \times \mathbf{x})\ \boldsymbol{\mu}\mathbf{n} \\ &= \int_{\partial\mathcal{Z}} \boldsymbol{\mu}\,\boldsymbol{\omega}\,\mathbf{x}\,[[\mathbf{Y}]]\mathbf{n}\ \boldsymbol{\mu}\mathbf{n} = \omega \int_{\partial\mathcal{Z}} \boldsymbol{\mu}\,\mathbf{k}\,\mathbf{x}\,[[\mathbf{Y}]]\mathbf{n}\ \boldsymbol{\mu}\mathbf{n}\,.\end{aligned}$$

Writing the dissipation $\mathcal{D}$ as $M_0\,\omega$, the driving couple M_0 is:

$$M_0 = \int_{\partial\mathcal{Z}} \boldsymbol{\mu}\,\mathbf{k}\,\mathbf{x}\,[[\mathbf{Y}]]\mathbf{n}\;\boldsymbol{\mu}\mathbf{n}\,.$$

3.3 Homothetically transforming defect

For a homothetically transforming defect with velocity field $\dot{\boldsymbol{\chi}}(\mathbf{x}) = \alpha\mathbf{x} \in \mathcal{Z}$, in which α is a scalar, the dissipation is given by:

$$\mathcal{D} = \int_{\partial\mathcal{Z}} \mathbf{g}([[\mathbf{Y}]]\mathbf{n}, \dot{\boldsymbol{\chi}})\;\boldsymbol{\mu}\mathbf{n} = \alpha\int_{\partial\mathcal{Z}} \mathbf{g}([[\mathbf{Y}]]\mathbf{n}, \mathbf{x})\;\boldsymbol{\mu}\mathbf{n}\,.$$

4 INVARIANCE CONDITIONS

Let us summarize hereafter some preliminary results that will be referred to in the sequel.

Proposition 4.1 *For any domain $\mathcal{C}$ of a reference placement $\mathbb{M}$, in absence of body forces (i.e. $\operatorname{div}\mathbf{P} = 0$) we have that:*

$$\int_{\partial\mathcal{C}} (\mathbf{Pn})(\mathbf{x}) \times \boldsymbol{\varphi}(\mathbf{x})\;\boldsymbol{\mu}\mathbf{n} = 0\,.$$

Proof. By the divergence theorem and subsequent integration by parts, we have that:

$$\int_{\partial\mathcal{C}} (\mathbf{Pn})(\mathbf{x}) \times \boldsymbol{\varphi}(\mathbf{x})\;\boldsymbol{\mu}\mathbf{n} = \int_{\mathcal{C}} (\operatorname{div}\mathbf{P})(\mathbf{x}) \times \boldsymbol{\varphi}(\mathbf{x})\;\boldsymbol{\mu} + \int_{\mathcal{C}} \operatorname{axial}\operatorname{skew}(\mathbf{P}d\boldsymbol{\varphi}^T)(\mathbf{x})\;\boldsymbol{\mu}\,.$$

By the symmetry of the tensor $\mathbf{P}d\boldsymbol{\varphi}^T = d\boldsymbol{\varphi}\mathbf{S}d\boldsymbol{\varphi}^T$ the last integral vanishes. Moreover, being $\operatorname{div}\mathbf{P} = 0$, the first integral at the r.h.s. vanishes too and the result is proven.

Proposition 4.2 *In an isotropic elastic medium $\mathbb{M}$, for any domain $\mathcal{C} \subset \mathbb{M}$ in which $\operatorname{div}\mathbf{Y} = 0$, we have:*

$$\int_{\partial\mathcal{C}} (\mathbf{Yn})(\mathbf{x}) \times \mathbf{x}\;\boldsymbol{\mu}\mathbf{n} = \int_{\partial\mathcal{C}} (W\mathbf{n})(\mathbf{x}) \times \mathbf{x}\;\boldsymbol{\mu}\mathbf{n} - \int_{\partial\mathcal{C}} (d\boldsymbol{\varphi}^T\mathbf{Pn})(\mathbf{x}) \times \mathbf{x}\;\boldsymbol{\mu}\mathbf{n} = 0\,.$$

Proof. By the divergence theorem and subsequent integration by parts, we have that:

$$\int_{\partial\mathcal{C}} (\mathbf{Yn})(\mathbf{x}) \times \mathbf{x}\;\boldsymbol{\mu}\mathbf{n} = \int_{\mathcal{C}} (\operatorname{div}\mathbf{Y})(\mathbf{x}) \times \mathbf{x}\;\boldsymbol{\mu} + \int_{\mathcal{C}} \operatorname{axial}\operatorname{skew}\mathbf{Y}(\mathbf{x})\;\boldsymbol{\mu}\,.$$

At the r.h.s. the first integral vanishes being by assumption $\operatorname{div}\mathbf{Y} = 0$ and the second integral vanishes since in a isotropic elastic medium $\operatorname{skew}\mathbf{Y} = 0$ by Lemma 6.2 in Appendix.

By propositions 4.1 and 4.2 we deduce (Knowles and Sternberg, 1972):

Proposition 4.3 *In an isotropic elastic medium $\mathbb{M}$, for any domain $\mathcal{C} \subset \mathbb{M}$ in which $\operatorname{div}\mathbf{Y} = 0$ and $\operatorname{div}\mathbf{P} = 0$ (absence of body forces), we have:*

$$\int_{\partial\mathcal{C}} (\mathbf{Yn})(\mathbf{x}) \times \mathbf{x}\;\boldsymbol{\mu}\mathbf{n} + \int_{\partial\mathcal{C}} (\mathbf{Pn})(\mathbf{x}) \times \boldsymbol{\varphi}(\mathbf{x})\;\boldsymbol{\mu}\mathbf{n} = \int_{\partial\mathcal{C}} \big((W\mathbf{n})(\mathbf{x}) \times \mathbf{x} - (d\boldsymbol{\varphi}^T\mathbf{Pn})(\mathbf{x}) \times \mathbf{x} + (\mathbf{Pn})(\mathbf{x}) \times \boldsymbol{\varphi}(\mathbf{x})\big)\;\boldsymbol{\mu}\mathbf{n} = 0\,.$$

Proposition 4.4 *In an elastic medium $\mathbb{M}$, for any domain $\mathcal{C} \subset \mathbb{M}$ in which $\operatorname{div}\mathbf{Y} = 0$ and $\operatorname{div}\mathbf{P} = 0$ (absence of body forces), we have:*

$$\int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Yn}, \mathbf{x})\;\boldsymbol{\mu}\mathbf{n} + \int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Pn}, \boldsymbol{\varphi}(\mathbf{x}))\;\boldsymbol{\mu}\mathbf{n} = 3\int_{\mathcal{C}} W\;\boldsymbol{\mu}\,.$$

Proof. By the divergence theorem and subsequent integration by parts, we have that:

$$\int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Yn}, \mathbf{x})\;\boldsymbol{\mu}\mathbf{n} = \int_{\mathcal{C}} \mathbf{g}(\operatorname{div}\mathbf{Y}, \mathbf{x})\;\boldsymbol{\mu} + \int_{\mathcal{C}} \langle \mathbf{Y}, \mathbf{I}\rangle_{\mathbf{g}}\;\boldsymbol{\mu}\,.$$

At the r.h.s. the first integral vanishes being by assumption $\operatorname{div}\mathbf{Y} = 0$. Substituting the expression $\mathbf{Y} = W\mathbf{I} - d\boldsymbol{\varphi}^T\mathbf{P}$ in the second integral and observing that $\langle W\mathbf{I}, \mathbf{I}\rangle_{\mathbf{g}} = 3\,W$, we get:

$$\int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Yn}, \mathbf{x})\;\boldsymbol{\mu}\mathbf{n} = \int_{\mathcal{C}} \langle W\mathbf{I}, \mathbf{I}\rangle_{\mathbf{g}}\;\boldsymbol{\mu} - \int_{\mathcal{C}} \langle d\boldsymbol{\varphi}^T\mathbf{P}, \mathbf{I}\rangle_{\mathbf{g}}\;\boldsymbol{\mu} = 3\int_{\mathcal{C}} W\;\boldsymbol{\mu} - \int_{\mathcal{C}} \langle \mathbf{P}, d\boldsymbol{\varphi}\rangle_{\mathbf{g}}\;\boldsymbol{\mu}\,.$$

Moreover, being $\operatorname{div}\mathbf{P} = 0$ we have that:

$$-\langle \mathbf{P}, d\boldsymbol{\varphi}\rangle_{\mathbf{g}} = \mathbf{g}(\operatorname{div}\mathbf{P}, \boldsymbol{\varphi}(\mathbf{x})) - \operatorname{div}(\mathbf{P}^T\boldsymbol{\varphi}(\mathbf{x})) = -\operatorname{div}(\mathbf{P}^T\boldsymbol{\varphi}(\mathbf{x}))\,,$$

and hence:

$$\begin{aligned}\int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Yn}, \mathbf{x})\;\boldsymbol{\mu}\mathbf{n} &= 3\int_{\mathcal{C}} W\;\boldsymbol{\mu} - \int_{\mathcal{C}} \langle \mathbf{P}, d\boldsymbol{\varphi}\rangle_{\mathbf{g}}\;\boldsymbol{\mu} \\ &= 3\int_{\mathcal{C}} W\;\boldsymbol{\mu} - \int_{\mathcal{C}} \operatorname{div}(\mathbf{P}^T\boldsymbol{\varphi}(\mathbf{x}))\;\boldsymbol{\mu} \\ &= 3\int_{\mathcal{C}} W\;\boldsymbol{\mu} - \int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{P}^T\boldsymbol{\varphi}(\mathbf{x}), \mathbf{n})\;\boldsymbol{\mu}\mathbf{n} \\ &= 3\int_{\mathcal{C}} W\;\boldsymbol{\mu} - \int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Pn}, \boldsymbol{\varphi}(\mathbf{x}))\;\boldsymbol{\mu}\mathbf{n}\,.\end{aligned}$$

This result is in agreement with the formula 2.10 reported in (Green, 1973).

4.1 Small displacement formulation

Many engineering applications can be dealt with by a geometrically linearized formulation. To specialize the previous invariance conditions to this important class of problems, it is convenient to reformulate the analysis in terms of the displacement field $\mathbf{u} \in \mathrm{C}^0(\mathbb{M};\mathbb{TS}) \cap \mathrm{C}^1(\mathcal{T}(\mathbb{M});\mathbb{TS})$ defined by $\mathbf{u}(\mathbf{x}) := \boldsymbol{\varphi}(\mathbf{x}) - \mathbf{x}$, so that $d\mathbf{u} = d\boldsymbol{\varphi} - \mathbf{I}$ in $\mathcal{T}(\mathbb{M})$. For the jump across the phase-transition interfaces $\mathcal{I}$ we have the equality $[[d\mathbf{u}]] = [[d\boldsymbol{\varphi}]]$ and hence the Eshelby's stress tensor can be equivalently defined in terms of displacement field as

$$\mathbf{Y_u} := W\mathbf{I} - d\mathbf{u}^T\mathbf{P} = W\mathbf{I} - d\boldsymbol{\varphi}^T\mathbf{P} + \mathbf{P} = \mathbf{Y} + \mathbf{P},$$

with $[[\mathbf{Y_u}]]\mathbf{n} = [[\mathbf{Y}]]\mathbf{n}$ since $[[\mathbf{Pn}]] = 0$. In the geometrically linearized theory, the reference and the actual placements of the body are taken to be coincident so that the Piola stress $\mathbf{P}$ and the Cauchy stress $\mathbf{T}$ may be identified. Accordingly the Eshelby's energy-momentum tensor takes the form

$$\mathbf{Y_u} = W\mathbf{I} - d\mathbf{u}^T\mathbf{T},$$

and its divergence is given by (Romano and Barretta, 2007):

$$\mathbf{g}(\operatorname{div}\mathbf{Y_u}, \mathbf{h}) = \mathbf{g}(d_3W, \mathbf{h}) - \langle \mathbf{S}_{\mathbb{M}}, d_{\mathbf{h}}\boldsymbol{\Delta} \rangle_{\mathbf{g}} + \mathbf{g}(\mathbf{b}, d_{\mathbf{h}}\mathbf{u}).$$

Proposition 4.5 *In an isotropic elastic medium $\mathbb{M}$, for any domain $\mathcal{C} \subset \mathbb{M}$ in which $\operatorname{div}\mathbf{Y_u} = 0$ and $\operatorname{div}\mathbf{T} = 0$ (absence of body forces), we have:*

$$\int_{\partial\mathcal{C}} (\mathbf{Y_u n})(\mathbf{x}) \times \mathbf{x}\ \boldsymbol{\mu}\mathbf{n} + \int_{\partial\mathcal{C}} (\mathbf{Tn})(\mathbf{x}) \times \mathbf{u}(\mathbf{x})\ \boldsymbol{\mu}\mathbf{n} =$$
$$\int_{\partial\mathcal{C}} \big((W\mathbf{n})(\mathbf{x}) \times \mathbf{x} - (d\mathbf{u}^T\mathbf{Tn})(\mathbf{x}) \times \mathbf{x}$$
$$+ (\mathbf{Tn})(\mathbf{x}) \times \mathbf{u}(\mathbf{x})\big)\ \boldsymbol{\mu}\mathbf{n} = 0.$$

Proof. Being $d\boldsymbol{\varphi} = d\mathbf{u} + \mathbf{I}$ and $\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \mathbf{x}$, by proposition 4.3 a direct computation proves the statement:

$$\int_{\partial\mathcal{C}} \big((W\mathbf{n})(\mathbf{x}) \times \mathbf{x} - (d\boldsymbol{\varphi}^T\mathbf{Pn})(\mathbf{x}) \times \mathbf{x}$$
$$+ (\mathbf{Pn})(\mathbf{x}) \times \boldsymbol{\varphi}(\mathbf{x})\big)\ \boldsymbol{\mu}\mathbf{n} =$$
$$\int_{\partial\mathcal{C}} \big((W\mathbf{n})(\mathbf{x}) \times \mathbf{x} - (d\mathbf{u}^T\mathbf{Tn})(\mathbf{x}) \times \mathbf{x} - (\mathbf{Tn})(\mathbf{x}) \times \mathbf{x} +$$
$$(\mathbf{Tn})(\mathbf{x}) \times \mathbf{u}(\mathbf{x}) + (\mathbf{Tn})(\mathbf{x}) \times \mathbf{x}\big)\ \boldsymbol{\mu}\mathbf{n} =$$
$$\int_{\partial\mathcal{C}} \big((W\mathbf{n})(\mathbf{x}) \times \mathbf{x} - (d\mathbf{u}^T\mathbf{Tn})(\mathbf{x}) \times \mathbf{x}$$
$$+ (\mathbf{Tn})(\mathbf{x}) \times \mathbf{u}(\mathbf{x})\big)\ \boldsymbol{\mu}\mathbf{n} = 0 \qquad \forall \mathcal{C} \subset \mathbb{M}.$$

This result is in agreement with the formula 2b reported in (Budiansky and Rice, 1973) on page 202.

Proposition 4.6 *In a linearly elastic medium $\mathbb{M}$, for any domain $\mathcal{C} \subset \mathbb{M}$ in which $\operatorname{div}\mathbf{Y_u} = 0$ and $\operatorname{div}\mathbf{T} = 0$ (absence of body forces), we have:*

$$\int_{\partial\mathcal{C}} \big(\mathbf{g}(\mathbf{Y_u n}, \mathbf{x}) - \frac{1}{2}\mathbf{g}(\mathbf{Pn}, \mathbf{u})\big)\ \boldsymbol{\mu}\mathbf{n} = 0.$$

Proof. Being $d\boldsymbol{\varphi} = d\mathbf{u} + \mathbf{I}$ and $\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \mathbf{x}$, a direct computation shows that:

$$\begin{aligned}\int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Yn}, \mathbf{x})\ \boldsymbol{\mu}\mathbf{n} &= \int_{\partial\mathcal{C}} \mathbf{g}(W\mathbf{n}, \mathbf{x})\ \boldsymbol{\mu}\mathbf{n} \\ &\quad - \int_{\partial\mathcal{C}} \mathbf{g}(d\boldsymbol{\varphi}^T\mathbf{Pn}, \mathbf{x})\ \boldsymbol{\mu}\mathbf{n} \\ &= \int_{\partial\mathcal{C}} \mathbf{g}(W\mathbf{n}, \mathbf{x})\ \boldsymbol{\mu}\mathbf{n} \\ &\quad - \int_{\partial\mathcal{C}} \mathbf{g}(d\mathbf{u}^T\mathbf{Pn}, \mathbf{x})\ \boldsymbol{\mu}\mathbf{n} \\ &\quad - \int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Pn}, \mathbf{x})\ \boldsymbol{\mu}\mathbf{n},\end{aligned}$$

$$\begin{aligned}\int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Pn}, \boldsymbol{\varphi}(\mathbf{x}))\ \boldsymbol{\mu}\mathbf{n} &= \int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Pn}, \mathbf{x})\ \boldsymbol{\mu}\mathbf{n} \\ &\quad + \int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Pn}, \mathbf{u})\ \boldsymbol{\mu}\mathbf{n}.\end{aligned}$$

Moreover by Clapeyron's theorem and in absence of body force we have:

$$3\int_{\mathcal{C}} W\ \boldsymbol{\mu} = \frac{3}{2}\int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Pn}, \mathbf{u})\ \boldsymbol{\mu}\mathbf{n}.$$

Finally, by lemma 4.4 the relation

$$\int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Yn}, \mathbf{x})\ \boldsymbol{\mu}\mathbf{n} + \int_{\partial\mathcal{C}} \mathbf{g}(\mathbf{Pn}, \boldsymbol{\varphi}(\mathbf{x}))\ \boldsymbol{\mu}\mathbf{n} = 3\int_{\mathcal{C}} W\ \boldsymbol{\mu}$$

becomes:

$$\int_{\partial\mathcal{C}} \big(\mathbf{g}(\mathbf{Y_u n}, \mathbf{x}) - \frac{1}{2}\mathbf{g}(\mathbf{Tn}, \mathbf{u})\big)\ \boldsymbol{\mu}\mathbf{n} = 0.$$

The result is in agreement with the formula 3.19 of Knowles and Sternberg reported in (Knowles and Sternberg, 1972) on page 198.

Remark 4.1 *Let us consider a closed surface $\mathcal{S} \subset \mathbb{M}$ enclosing a defect $\mathcal{Z}$ and let C be the domain whose boundary is $\partial C = \mathcal{S} \cup \partial\mathcal{Z}$. Assuming that $\operatorname{div}\mathbf{Y_u} = 0$ in C, by the divergence theorem it follows that the integral*

$$\mathbf{J} = \int_{\mathcal{S}} \big((W\mathbf{n})(\mathbf{x}) - (d\mathbf{u}^T\mathbf{Tn})(\mathbf{x})\big)\ \boldsymbol{\mu}\mathbf{n}$$

is independent of $\mathcal{S}$*. Moreover, if* $\operatorname{div}\mathbf{T}=0$ *(absence of body forces) in* $\mathcal{C}$*, then*

$$M=\int_{\mathcal{S}}\big(\mathbf{g}((W\mathbf{n})(\mathbf{x}),\mathbf{x})-\mathbf{g}((d\mathbf{u}^T\mathbf{Tn})(\mathbf{x}),\mathbf{x})-\frac{1}{2}\mathbf{g}((\mathbf{Pn})(\mathbf{x}),\mathbf{u}(\mathbf{x}))\big)\ \boldsymbol{\mu}\mathbf{n}$$

is independent of $\mathcal{S}$ *(see proposition 4.6). The further assumption of an isotropic medium* $\mathbb{M}$*, leads to the natural definition of the* $\mathcal{S}$*-invariant integral (see proposition 4.5):*

$$\mathbf{L}=\int_{\mathcal{S}}\big((W\mathbf{n})(\mathbf{x})\times\mathbf{x}-(d\mathbf{u}^T\mathbf{Tn})(\mathbf{x})\times\mathbf{x}+(\mathbf{Tn})(\mathbf{x})\times\mathbf{u}(\mathbf{x})\big)\ \boldsymbol{\mu}\mathbf{n}\,.$$

These integrals are known in the literature as conservation theorems (Knowles and Sternberg, 1972), (Budiansky and Rice, 1973). Their validity is subject to the assumptions of a homogeneous elastic medium, under homogeneous anelastic deformation and no body forces (i.e. $\operatorname{div}\mathbf{Y}_{\mathbf{u}}=0$ *(Romano et al., 2005)). In the next section we remove the hypothesis that* $\operatorname{div}\mathbf{Y}_{\mathbf{u}}=0$ *in* $\mathbb{M}$ *and further invariance conditions for the evaluation of the dissipation associated with a transforming defect are provided.*

5 SPECIAL INSTANCES

Example 1

Let us consider a crack-like defect $\mathcal{Z}$ translating in an elastic medium $\mathbb{M}$ characterized by the property:

$$(\operatorname{div}\mathbf{Y})(\mathbf{x})=0\quad\forall\,\mathbf{x}\in\mathcal{Z}\,.$$

Then, being

$$\int_{\partial\mathcal{Z}}\mathbf{g}(\mathbf{Y}^-\mathbf{n},\mathbf{d})\ \boldsymbol{\mu}\mathbf{n}=\int_{\mathcal{Z}}\mathbf{g}(\operatorname{div}\mathbf{Y},\mathbf{d})\ \boldsymbol{\mu}=0\,,$$

the driving force (see subsection 3.1) is given by:

$$\begin{aligned}F&=\int_{\partial\mathcal{Z}}\mathbf{g}([[\mathbf{Y}]]\mathbf{n},\mathbf{d})\ \boldsymbol{\mu}\mathbf{n}\\&=\int_{\partial\mathcal{Z}}\mathbf{g}(\mathbf{Y}^+\mathbf{n},\mathbf{d})\ \boldsymbol{\mu}\mathbf{n}-\int_{\partial\mathcal{Z}}\mathbf{g}(\mathbf{Y}^-\mathbf{n},\mathbf{d})\ \boldsymbol{\mu}\mathbf{n}\\&=\int_{\partial\mathcal{Z}}\mathbf{g}(W^+\mathbf{n},\mathbf{d})\ \boldsymbol{\mu}\mathbf{n}-\int_{\partial\mathcal{Z}}\mathbf{g}(\mathbf{Tn},d_{\mathbf{d}}\mathbf{u}^+)\ \boldsymbol{\mu}\mathbf{n}\,.\end{aligned}$$

The following result then holds:

Proposition 5.1 *In an elastic medium* $\mathbb{M}$*, for any closed surface* Σ *enclosing a translating defect* $\mathcal{Z}\subset\mathbb{M}$ *and* n *elastic phases* $\mathcal{P}_i\subset\mathbb{M}$ *(Fig. 2), the driving force acting on* $\mathcal{Z}$ *is given by:*

$$\begin{aligned}F&=J(\Sigma)-\sum_{i=1}^{n}\int_{\partial\mathcal{P}_i}\mathbf{g}(\mathbf{Yn},\mathbf{d})\ \boldsymbol{\mu}\mathbf{n}-\int_{C(\Sigma)}\mathbf{g}(\operatorname{div}\mathbf{Y},\mathbf{d})\ \boldsymbol{\mu}\\&=J(\Sigma)-\sum_{i=1}^{n}\int_{\partial\mathcal{P}_i}\mathbf{g}(\mathbf{Yn},\mathbf{d})\ \boldsymbol{\mu}\mathbf{n}-\int_{C(\Sigma)}\mathbf{g}(d_3W,\mathbf{d})\,\boldsymbol{\mu}\\&\quad+\int_{C(\Sigma)}\langle\mathbf{T_a},d_{\mathbf{d}}\boldsymbol{\Delta}\rangle_{\mathbf{g}}\,\boldsymbol{\mu}-\int_{C(\Sigma)}\mathbf{g}(\mathbf{b},d_{\mathbf{d}}\mathbf{u})\,\boldsymbol{\mu}\,,\end{aligned}$$

where $C(\Sigma)$ *is a domain of* $\mathbb{M}$ *whose boundary* $\partial C(\Sigma)=\partial\mathcal{Z}\cup\Sigma\cup_{i=1}^{n}\partial\mathcal{P}_i$ *and*

$$J(\Sigma):=\int_{\Sigma}\mathbf{g}(\mathbf{Yn},\mathbf{d})\ \boldsymbol{\mu}\mathbf{n}$$

is the J*-integral (Rice, 1968) associated with the surface* Σ *(it is the projection along the unit vector* $\mathbf{d}$ *of the vector integral* $\mathbf{J}$ *defined in remark 4.1):*

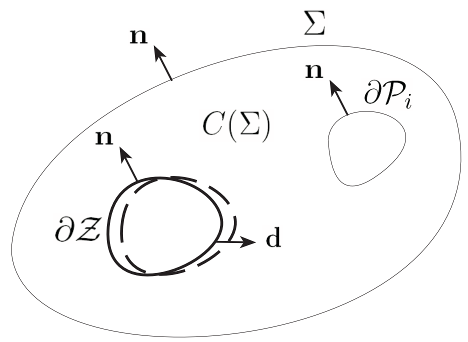

Figure 2. Translating defect.

Proof. The divergence theorem and the formula for $\operatorname{div}\mathbf{Y}$ of section 2, give:

$$\begin{aligned}\int_{C(\Sigma)}\mathbf{g}(\operatorname{div}\mathbf{Y},\mathbf{d})\ \boldsymbol{\mu}&=\int_{\partial C(\Sigma)}\mathbf{g}(\mathbf{Yn},\mathbf{d})\ \boldsymbol{\mu}\mathbf{n}\\&=-F+J(\Sigma)-\sum_{i=1}^{n}\int_{\partial\mathcal{P}_i}\mathbf{g}(\mathbf{Yn},\mathbf{d})\ \boldsymbol{\mu}\mathbf{n}\\&=\int_{C(\Sigma)}\mathbf{g}(d_3W,\mathbf{d})\,\boldsymbol{\mu}\\&\quad-\int_{C(\Sigma)}\langle\mathbf{T_a},d_{\mathbf{d}}\boldsymbol{\Delta}\rangle_{\mathbf{g}}\,\boldsymbol{\mu}+\int_{C(\Sigma)}\mathbf{g}(\mathbf{b},d_{\mathbf{d}}\mathbf{u})\,\boldsymbol{\mu}\,,\end{aligned}$$

and the result follows.

Remark 5.1 *By considering a homogeneous elastic domain* $C(\Sigma)$*, under homogeneous anelastic deformation and no body forces (i.e.* $\operatorname{div}\mathbf{Y}=0$*) the formula of proposition 5.5 becomes:*

$$F=J(\Sigma)-\sum_{i=1}^{n}\int_{\partial\mathcal{P}_i}\mathbf{g}(\mathbf{Yn},\mathbf{d})\;\boldsymbol{\mu}\mathbf{n}\,.$$

This result is in agreement with the formula provided by Kikuchi and Miyamoto (Kikuchi and Miyamoto, 1982) and quoted by Brocks and Scheider (Brocks and Scheider, 2001) in establishing a Σ*-independence condition for the evaluation of the driving force associated with defects propagating in composite materials or in welded structures.*

Example 2

Let us consider a defect $\mathcal{Z}$ rotating in an isotropic elastic medium $\mathbb{M}$ characterized by the property:

$$(\operatorname{div}\mathbf{Y})(\mathbf{x})=0\quad\forall\mathbf{x}\in\mathcal{Z}\,.$$

Then, by proposition 4.2 being

$$\int_{\partial\mathcal{Z}}(\mathbf{Y}^{-}\mathbf{n})(\mathbf{x})\times\mathbf{x}\;\boldsymbol{\mu}\mathbf{n}=0\,,$$

the driving couple (see subsection 3.2) is given by:

$$\begin{aligned}M_0&=\int_{\partial\mathcal{Z}}\boldsymbol{\mu}\,\mathbf{k}\,\mathbf{x}\,[[\mathbf{Y}]]\mathbf{n}\;\boldsymbol{\mu}\mathbf{n}=-\int_{\partial\mathcal{Z}}\boldsymbol{\mu}\,([[\mathbf{Y}]]\mathbf{n})\mathbf{x}\mathbf{k}\;\boldsymbol{\mu}\mathbf{n}\\&=-\int_{\partial\mathcal{Z}}\mathbf{g}([[\mathbf{Y}]]\mathbf{n}\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\mathbf{n}\\&=\int_{\partial\mathcal{Z}}\mathbf{g}(\mathbf{Y}^{-}\mathbf{n}\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\mathbf{n}-\int_{\partial\mathcal{Z}}\mathbf{g}(\mathbf{Y}^{+}\mathbf{n}\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\mathbf{n}\\&=-\int_{\partial\mathcal{Z}}\mathbf{g}(\mathbf{Y}^{+}\mathbf{n}\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\mathbf{n}\,.\end{aligned}$$

By removing the assumption that $\operatorname{div}\mathbf{Y}=0$ in proposition 4.2, we have that:

Proposition 5.2 *In an isotropic elastic medium in its reference configuration* $\mathbb{M}$*, for any domain* $\mathcal{C}\subset\mathbb{M}$*, the following invariance condition holds:*

$$\int_{\partial\mathcal{C}}(\mathbf{Yn})(\mathbf{x})\times\mathbf{x}\;\boldsymbol{\mu}\mathbf{n}-\int_{\mathcal{C}}(\operatorname{div}\mathbf{Y})(\mathbf{x})\times\mathbf{x}\;\boldsymbol{\mu}=0\,.$$

A further result is the following:

Proposition 5.3 *In an isotropic elastic medium* $\mathbb{M}$*, for any closed surface* Σ *enclosing a rotating defect* $\mathcal{Z}\subset\mathbb{M}$ *and* n *elastic phases* $\mathcal{P}_i\subset\mathbb{M}$ *(Fig. 3), the driving couple acting on* $\mathcal{Z}$ *is given by:*

$$\begin{aligned}M_0&=-B(\Sigma)+\sum_{i=1}^{n}\int_{\partial\mathcal{P}_i}\mathbf{g}((\mathbf{Y}^{+}\mathbf{n})(\mathbf{x})\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\mathbf{n}\\&\quad+\int_{C(\Sigma)}\mathbf{g}((\operatorname{div}\mathbf{Y})(\mathbf{x})\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\,,\end{aligned}$$

where $C(\Sigma)$ *is a domain of* $\mathbb{M}$ *whose boundary* $\partial C(\Sigma)=\partial\mathcal{Z}\cup\Sigma\cup_{i=1}^{n}\partial\mathcal{P}_i$ *and*

$$B(\Sigma):=\int_{\Sigma}\mathbf{g}((\mathbf{Yn})(\mathbf{x})\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\mathbf{n}$$

is the B*-integral associated with the surface* Σ *(this definition naturally follows by proposition 4.2).*

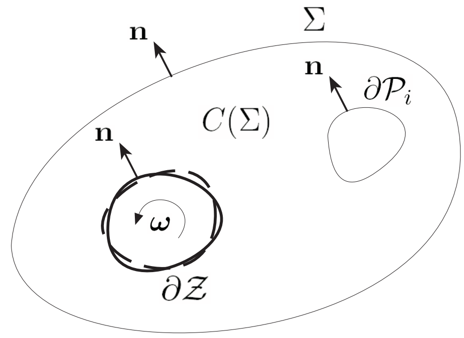

Figure 3. Rotating defect.

Proof. Being $\partial C(\Sigma)=\partial\mathcal{Z}\cup\Sigma\cup_{i=1}^{n}\partial\mathcal{P}_i$, the proposition 5.2 provides:

$$\begin{aligned}&-\int_{\partial\mathcal{Z}}\mathbf{g}((\mathbf{Y}^{+}\mathbf{n})(\mathbf{x})\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\mathbf{n}\\&+\int_{\Sigma}\mathbf{g}((\mathbf{Yn})(\mathbf{x})\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\mathbf{n}\\&-\sum_{i=1}^{n}\int_{\partial\mathcal{P}_i}\mathbf{g}((\mathbf{Y}^{+}\mathbf{n})(\mathbf{x})\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\mathbf{n}\\&-\int_{C(\Sigma)}\mathbf{g}((\operatorname{div}\mathbf{Y})(\mathbf{x})\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}=0\,,\end{aligned}$$

and obseving that:

$$\begin{aligned}M_0&=-\int_{\partial\mathcal{Z}}\mathbf{g}(\mathbf{Y}^{+}\mathbf{n}\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\mathbf{n}\,,\\B(\Sigma)&=\int_{\Sigma}\mathbf{g}((\mathbf{Yn})(\mathbf{x})\times\mathbf{x},\mathbf{k})\;\boldsymbol{\mu}\mathbf{n}\,,\end{aligned}$$

the result follows.

Example 3

Let us consider a defect $\mathcal{Z}$ homothetically transforming in an elastic medium $\mathbb{M}$ characterized by the property:

$$(\operatorname{div}\mathbf{Y})(\mathbf{x})=0\quad\forall\mathbf{x}\in\mathcal{Z}\,.$$

Then, by proposition 4.4 being

$$\int_{\partial\mathcal{Z}}\mathbf{g}(\mathbf{Y}^{-}\mathbf{n},\mathbf{x})\;\boldsymbol{\mu}\mathbf{n}=3\int_{\mathcal{Z}}W\;\boldsymbol{\mu}-\int_{\partial\mathcal{Z}}\mathbf{g}(\mathbf{Pn},\varphi(\mathbf{x}))\;\boldsymbol{\mu}\mathbf{n}\,.$$

the dissipation (see subsection 3.3) is given by:

$$\begin{aligned}\mathcal{D} &= \alpha \int_{\partial \mathcal{Z}} \mathbf{g}([[\mathbf{Y}]]\mathbf{n}, \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} \\ &= \alpha \int_{\partial \mathcal{Z}} \mathbf{g}(\mathbf{Y}^+\mathbf{n}, \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} - \alpha \int_{\partial \mathcal{Z}} \mathbf{g}(\mathbf{Y}^-\mathbf{n}, \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} \\ &= \alpha \int_{\partial \mathcal{Z}} \mathbf{g}(\mathbf{Y}^+\mathbf{n}, \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} \\ &\quad -3\alpha \int_{\mathcal{Z}} W \ \boldsymbol{\mu} + \alpha \int_{\partial \mathcal{Z}} \mathbf{g}(\mathbf{P}\mathbf{n}, \boldsymbol{\varphi}(\mathbf{x})) \ \boldsymbol{\mu}\mathbf{n} \,.\end{aligned}$$

By removing the assumption that $\operatorname{div} \mathbf{Y} = 0$ in proposition 4.4 we have that:

Proposition 5.4 *Let us consider an elastic medium* $\mathbb{M}$. *In absence of body force (i.e.* $\operatorname{div} \mathbf{P} = 0$*) in* $\mathbb{M}$ *then for any domain* $\mathcal{C} \subset \mathbb{M}$ *we have:*

$$\int_{\partial \mathcal{C}} \mathbf{g}(\mathbf{Y}\mathbf{n}, \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} = 3 \int_{\mathcal{C}} W \ \boldsymbol{\mu} + \int_{\mathcal{C}} \mathbf{g}(\operatorname{div} \mathbf{Y}, \mathbf{x}) \ \boldsymbol{\mu} - \int_{\partial \mathcal{C}} \mathbf{g}(\mathbf{P}\mathbf{n}, \boldsymbol{\varphi}(\mathbf{x})) \ \boldsymbol{\mu}\mathbf{n} \,.$$

A further result is the following:

Proposition 5.5 *In an elastic medium* $\mathbb{M}$, *for any closed surface* Σ *enclosing a homothetically transforming defect* $\mathcal{Z} \subset \mathbb{M}$ *and* n *elastic phases* $\mathcal{P}_i \subset \mathbb{M}$ *(Fig. 4), the dissipation* $\mathcal{D}$ *associated with the evolution of* $\mathcal{Z}$ *is given by* $\mathcal{D} = \alpha \overline{\mathcal{D}}$, *where:*

$$\begin{aligned}\overline{\mathcal{D}} &= \int_{\partial \mathcal{Z}} \mathbf{g}((\mathbf{Y}^+\mathbf{n})(\mathbf{x}), \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} \\ &\quad -3 \int_{\mathcal{Z}} W(\mathbf{x}) \ \boldsymbol{\mu} + \int_{\partial \mathcal{Z}} \mathbf{g}((\mathbf{P}\mathbf{n})(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{x})) \ \boldsymbol{\mu}\mathbf{n} \\ &= \int_{\Sigma} \mathbf{g}((\mathbf{Y}\mathbf{n})(\mathbf{x}), \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} \\ &\quad - \sum_{i=1}^{n} \int_{\partial \mathcal{P}_i} \mathbf{g}((\mathbf{Y}^+\mathbf{n})(\mathbf{x}), \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} \\ &\quad -3 \int_{C(\Sigma)} W(\mathbf{x}) \ \boldsymbol{\mu} - \int_{C(\Sigma)} \mathbf{g}((\operatorname{div} \mathbf{Y})(\mathbf{x}), \mathbf{x}) \ \boldsymbol{\mu} \\ &\quad - \int_{\Sigma} \mathbf{g}((\mathbf{P}\mathbf{n})(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{x})) \ \boldsymbol{\mu}\mathbf{n} \\ &\quad - \sum_{i=1}^{n} \int_{\partial \mathcal{P}_i} \mathbf{g}((\mathbf{P}\mathbf{n})(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{x})) \ \boldsymbol{\mu}\mathbf{n} \\ &\quad -3 \int_{\mathcal{Z}} W(\mathbf{x}) \ \boldsymbol{\mu} \,,\end{aligned}$$

in which $C(\Sigma)$ *is the domain of* $\mathbb{M}$ *whose boundary is* $\partial C(\Sigma) = \partial \mathcal{Z} \cup \Sigma \cup_{i=1}^{n} \partial \mathcal{P}_i$.

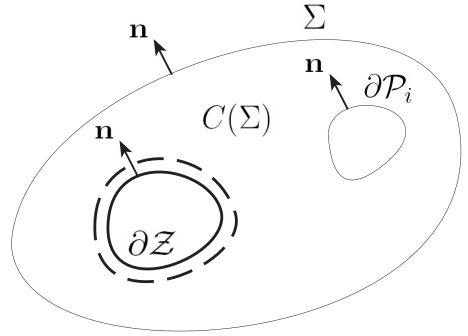

Figure 4. Homothetically transforming defect.

Proof. Being $\partial C(\Sigma) = \partial \mathcal{Z} \cup \Sigma \cup_{i=1}^{n} \partial \mathcal{P}_i$, the proposition 5.4 provides:

$$\begin{aligned}&- \int_{\partial \mathcal{Z}} \mathbf{g}((\mathbf{Y}^+\mathbf{n})(\mathbf{x}), \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} \\ &+ \int_{\Sigma} \mathbf{g}((\mathbf{Y}\mathbf{n})(\mathbf{x}), \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} - \sum_{i=1}^{n} \int_{\partial \mathcal{P}_i} \mathbf{g}((\mathbf{Y}^+\mathbf{n})(\mathbf{x}), \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} \\ &= 3 \int_{C(\Sigma)} W(\mathbf{x}) \ \boldsymbol{\mu} + \int_{C(\Sigma)} \mathbf{g}((\operatorname{div} \mathbf{Y})(\mathbf{x}), \mathbf{x}) \ \boldsymbol{\mu} \\ &- \Big(- \int_{\partial \mathcal{Z}} \mathbf{g}((\mathbf{P}\mathbf{n})(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{x})) \ \boldsymbol{\mu}\mathbf{n} \\ &+ \int_{\Sigma} \mathbf{g}((\mathbf{P}\mathbf{n})(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{x})) \ \boldsymbol{\mu}\mathbf{n} \\ &- \sum_{i=1}^{n} \int_{\partial \mathcal{P}_i} \mathbf{g}((\mathbf{P}\mathbf{n})(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{x})) \ \boldsymbol{\mu}\mathbf{n} \Big) \,,\end{aligned}$$

and observing that:

$$\overline{\mathcal{D}} = \int_{\partial \mathcal{Z}} \mathbf{g}((\mathbf{Y}^+\mathbf{n})(\mathbf{x}), \mathbf{x}) \ \boldsymbol{\mu}\mathbf{n} -3 \int_{\mathcal{Z}} W(\mathbf{x}) \ \boldsymbol{\mu} + \int_{\partial \mathcal{Z}} \mathbf{g}((\mathbf{P}\mathbf{n})(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{x})) \ \boldsymbol{\mu}\mathbf{n} \,,$$

the result follows.

6 APPENDIX

Lemma 6.1 *In an isotropic elastic medium the eigenspaces of the Piola-Kirchhoff stress* $\mathbf{S}$ *and of the Piola-Green operator* $\mathbf{D}(\boldsymbol{\varphi}) = d\boldsymbol{\varphi}^T d\boldsymbol{\varphi}$ *coincide.*

Proof. Let us consider an eigenvector $\mathbf{e} \in \mathbb{T}_{\mathbf{m}}\mathbb{M}$ of $\mathbf{D}(\boldsymbol{\varphi}) \in \mathrm{C}^1(\mathbb{T}_{\mathbf{m}}\mathbb{M}\,; \mathbb{T}_{\mathbf{m}}\mathbb{M})$ so that $\mathbf{D}(\boldsymbol{\varphi})\mathbf{e} = \lambda\, \mathbf{e}$ and let $\mathbf{R}_{\mathbf{e}} \in \operatorname{Orth}(\mathbb{T}_{\mathbf{m}}\mathbb{M}\,; \mathbb{T}_{\mathbf{m}}\mathbb{M})$ be the reflection across the plane perpendicular to $\mathbf{e}$. Setting $\boldsymbol{\Pi}_{\mathbf{e}} = \mathbf{I} - \mathbf{e} \otimes \mathbf{e}$, we have that

$$\mathbf{R}_{\mathbf{e}}\, \mathbf{e} = -\mathbf{e} \,, \qquad \mathbf{R}_{\mathbf{e}}\, \boldsymbol{\Pi}_{\mathbf{e}}\, \mathbf{e} = \boldsymbol{\Pi}_{\mathbf{e}}\, \mathbf{e} \,.$$

By the spectral decomposition of $\mathbf{D}(\boldsymbol{\varphi})$:

$$\mathbf{D}(\boldsymbol{\varphi}) = \sum_{i=1}^{3} \lambda_i\, \mathbf{e}_i \otimes \mathbf{e}_i \,,$$

with $\mathbf{e}_1 = \mathbf{e}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis of eigenvectors, follows that $\mathbf{R_e}\,\mathbf{D}(\boldsymbol{\varphi})\,\mathbf{R_e}^T = \mathbf{D}(\boldsymbol{\varphi})$ (i.e. $\mathbf{R_e}$ and $\mathbf{D}(\boldsymbol{\varphi})$ commute). Moreover, by assumption, being $\mathbf{S} = \partial W(\mathbf{D}(\boldsymbol{\varphi}))$ an isotropic law:

$$\mathbf{R_e}\,\partial W(\mathbf{D}(\boldsymbol{\varphi}))\,\mathbf{R_e}^T = \partial W(\mathbf{R_e}\,\mathbf{D}(\boldsymbol{\varphi})\,\mathbf{R_e}^T)$$

for any $\mathbf{D}(\boldsymbol{\varphi}) \in \mathrm{C}^1(\mathbb{T}_\mathbf{m}\mathbb{M}\,; \mathbb{T}_\mathbf{m}\mathbb{M})$ and for any $\mathbf{R_e} \in \mathrm{Orth}\,(\mathbb{T}_\mathbf{m}\mathbb{M}\,; \mathbb{T}_\mathbf{m}\mathbb{M})$, we infer that:

$$\mathbf{R_e}\,\partial W(\mathbf{D}(\boldsymbol{\varphi}))\,\mathbf{R_e}^T = \partial W(\mathbf{R_e}\,\mathbf{D}(\boldsymbol{\varphi})\,\mathbf{R_e}^T) = \partial W(\mathbf{D}(\boldsymbol{\varphi}))\,,$$

so that $\mathbf{R_e}$ commutes with $\mathbf{S} = \partial W(\mathbf{D}(\boldsymbol{\varphi}))$. Accordingly, since

$$\mathbf{R_e}\,\partial W(\mathbf{D}(\boldsymbol{\varphi}))\,\mathbf{e} = \partial W(\mathbf{D}(\boldsymbol{\varphi}))\mathbf{R_e}\,\mathbf{e} = -\partial W(\mathbf{D}(\boldsymbol{\varphi}))\,\mathbf{e}\,,$$

it follows that $\partial W(\mathbf{D}(\boldsymbol{\varphi}))\,\mathbf{e} = \alpha\,\mathbf{e}$, with $\alpha \in \mathcal{R}$. Therefore every eigenvector of $\mathbf{D}(\boldsymbol{\varphi})$ is an eigenvector of $\mathbf{S}$.

Lemma 6.2 *In an isotropic elastic medium the Eshelby stress tensor* $\mathbf{Y} = W\mathbf{I} - d\boldsymbol{\varphi}^T\mathbf{P}$ *is symmetric.*

Proof. By lemma 6.1 we write:

$$\mathbf{D}(\boldsymbol{\varphi})\,\mathbf{e} = \alpha\,\mathbf{e}\,, \qquad \mathbf{S}\mathbf{e} = \beta\,\mathbf{e}\,, \quad \alpha, \beta \in \mathcal{R}\,,$$

and a direct computation shows that

$$\mathbf{D}(\boldsymbol{\varphi})\,\mathbf{S}\,\mathbf{e} = \alpha\,\beta\,\mathbf{e}$$
$$\mathbf{S}\,\mathbf{D}(\boldsymbol{\varphi})\,\mathbf{e} = \alpha\,\beta\,\mathbf{e}\,.$$

Therefore the operators $\mathbf{S}$ and $\mathbf{D}(\boldsymbol{\varphi})$ commute:

$$\mathbf{D}(\boldsymbol{\varphi})\,\mathbf{S} = \mathbf{S}\,\mathbf{D}(\boldsymbol{\varphi})\,,$$

that is $d\boldsymbol{\varphi}^T d\boldsymbol{\varphi}\,\mathbf{S} = \mathbf{S}\,d\boldsymbol{\varphi}^T d\boldsymbol{\varphi}$. Being $\mathbf{P} = d\boldsymbol{\varphi}\,\mathbf{S}$ and $\mathbf{S} = \mathbf{S}^T$, we infer that $d\boldsymbol{\varphi}^T\mathbf{P}$ is symmetric.

7 CONCLUSIONS

The dissipation associated with the evolution of a translating, rotating or homothetically transforming defect in a multi phase material is evaluated. The analysis is carried out by simulating a defect as an abrupt change of elastic properties in a material and describing its evolution as an elastic phase transition phenomenon. Some conservation laws, due to Knowles and Sternberg, are suitably extended to include more general situations in which the inhomogeneity of the elastic properties or of the anelastic deformation leads to a non-vanishing divergence of the energy-momentum tensor.

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