ASSESSMENT OF MATERIALS NONLINEARITY IN FRAMED STRUCTURES OF REINFORCED CONCRETE AND COMPOSITES

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Abstract:
The assessment of the fatigue loading of a structure demands the consideration of the nonlinearity effects derived from the geometric variations and the bars stiffness changes. The resolution through matrixes methods frequently uses an incremental treatment for load application, being considered at each loading step, the solution research by means of an iterative process which result is expected to be convergent. In each increasing and iterations a variant of the equation of equilibrium can be brought up between loading and displacements differentials, that variant is solved in the linear field. The tangent stiffness matrix used has to consider an elastic and linear component and the other component that takes into account the geometric variations and mechanical nonlinearity of materials.

The present work deals with the component of the affected stiffness matrix due to the material mechanic nonlinearity. Its quantification in reinforced concrete and composites structures is especially complex because of the presence of several materials with different resistant behaviour each one and to nonlinearity between stresses and strains especially in the concrete case and to the inertia reduction of the section produced by concrete cracking.

The developing of an iterative process based on bisection method, which provides the equilibrium position of the section in front of known requests, taking into account the nonlinear constituent relationships of materials. Once the equilibrium has been obtained, the section stiffness modulus can be deduced from the expression $E{\cdot}I=M/\phi$.

1 INTRODUCTION

The frame structure design can be done under several hypotheses; currently the most important two of them are the assessment of geometric nonlinearity of the structure and the consideration of an elastic or plastic behaviour of materials. The interesting comparative study about different designing models made by Nethercot [1] can be consulted.

First approach to the structures design by using a matrix method was made by Livesley [2]. At this, the nodes deformations on framed structures are related to the loads applied by the named stiffness matrix. Nowadays the essence of his mathematical model is still preserved, and many of later advances in it are linked to the computer and software development.

Studies about the assessment of geometric nonlinearity of structure started with Jennings [3], being his work simplified by Yang and McGuire [4].

The approach to geometric nonlinear design is usually made by using an increasing set out of load application, using Zienkiewicz [5] terminology can be expressed by:

$$[K_L + K_G + K_0][\Delta u] = [\Delta F] \quad (1)$$

Where $[K_L]$ is the linear stiffness matrix, $[K_G]$ and $[K_0]$ are the matrixes that take into account, respectively, initial stresses and strains. $[\Delta u]$ and $[\Delta F]$ are the vectors that represent, at the same time, the increasing displacements and forces.

In contrast to geometric nonlinearity, the nonlinear behaviour of the materials is usually taken into account only for sections measurement. However, an accurate process must take it into consideration in the structure analysis process itself. In this case, the linear stiffness matrix $[K_L]$ of equation (1) is replaced by the named tangent matrix $[Kt]$.

The mechanical nonlinearity influences particularly at concrete structures, because of their elasticity module that depends on the load applied value and in the evaluation of the section inertia, the concrete cracking and the amount and reinforcing distribution must be taken into account.

From them, are the slender supports, the elements that in addition to the section strength, is especially necessary to take into account the second order effects and the nonlinear stress-strain behaviour of materials. There is a lot of studies about beam-column behaviour among them the ones made by Mavichak and Furlong [6], Al-Noury and Chen [7], Virdi and Dowling [8], Wang and Hsu [9] or Roik and Bergmann [10] must be quoted.

The present work develops an iterative procedure in order to obtain the stiffness modulus E·I of the section at each loading step taking into account the applied solicitations, the nonlinear constituent equations of materials, the concrete cracking and variation on modulus of elasticity.

This alternative procedure to the well-known quasi-Newton method proposed by Yen [11], is based on bisection method in order to get the section equilibrium. Once achieved this, the section modulus of stiffness can be deduced from the expression $E \cdot I = M/\phi$.

Subsequently this procedure has been implemented in a computer program about nonlinear analysis of structures that uses an increasing treatment in load application. The bar stiffness is corrected in each iteration for both reasons geometrical and mechanical of the materials.

In this way the assessment of mechanic nonlinearity of materials, although it affects compressed bars in a special way, this will make it applicable in the whole framed structure.

2 CALCULATION METHOD

2.1 Assumed hypothesis

Following hypothesis are accepted at the calculation process:

1. Plane surfaces remain plane after deformation (Bernoulli hypothesis). The deformation proportionally varies with the distance to the neutral fibre.
2. Between steel and concrete a compatibility of deformations exists in their contact surfaces.
3. Creep and shrinkage effects of concrete are neglected.
4. Effects produced by tangential stress are neglected.
5. Tensile strength of concrete and the stress increasing due to the steel hardening are neglected.

2.2 Constitutive laws of materials

The stress-strain relationships used in this work are provided by the Eurocodes regulations EC2 [12] and EC3 [13], which are based on relationships under observation experimentally tested in test tubes.

In the concrete case, the named parabola-rectangle diagram has been used, which relationship between stresses and strains can be expressed by the function:

$$
\begin{aligned}
&-0{,}002 < \varepsilon_c < 0\\
&\qquad \sigma_c = 1000\cdot\varepsilon_c\cdot(250\cdot\varepsilon_c+1)\\
&-0{,}035 \le \varepsilon_c \le -0{,}002\\
&\qquad \sigma_c = f_{ck}
\end{aligned} \tag{2}
$$

Maximum strain ε_{cu} taken in this diagram is about 0,35 %.

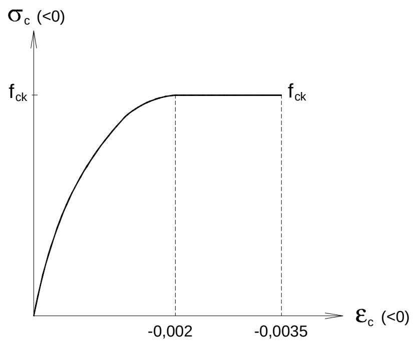

Figure 1 : Concrete stress-strain diagram

For steel case, a simplified stress-strain diagrams constituted by two branches will be adopted, first branch starts from the origin, with a slope equal to E_s (that is reached at 210kN/ mm^2) until f_{sk} (the specified value of the characteristic elastic limit, depending on the steel type).

A second horizontal branch will be adopted for a computer calculation with a slope of E_s / 10000, taking a maximum value of 1 % as the limit maximum of the unitary strain.

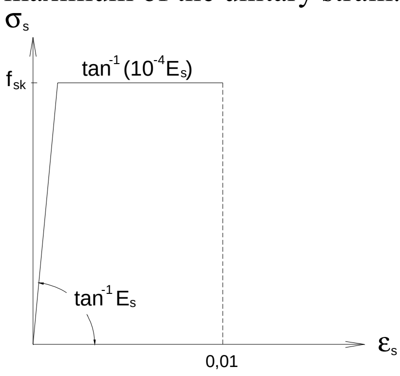

Figure 2 : Steel stress-strain diagram.

3 STRUCTURAL ANALYSIS

At structural design in elastic system, the equilibrium relationship is defined in a linear way between forces and displacements, where the matrix stiffness values of the structure are constant values. Its resolution consists on getting the displacements vector of nodes $\{\Delta\}$ from the relationship:

$$[K_e]\{\Delta\} = \{P\}$$

Where $[K_e]$ is the elastic and linear stiffness matrix and $\{P\}$ is the loads vector.

Main difference while dealing the second order effects consist on that the values that built up the structure stiffness matrix depend on displacements and the loads applied.

The solution research use to be brought up in an incremental way by the loads application. At each loading step it is solved by an iterative process, a variation of the equilibrium equation:

$$[K_t]\{d\Delta\} = \{dP\}$$

$[K_t]$ is the tangent stiffness matrix and $\{d\Delta\}$ and $\{dP\}$ respectively represent the vectors of both displacement and loads differentials. The tangent stiffness matrix is formed by a linear and elastic component $[K_e]$, and other nonlinear component, $[K_g]$ y $[K_m]$, according to McGuire terminology [12]. The first one, $[K_g]$, takes into consideration geometric variations of the bar and the second one $[K_m]$, considers the mechanical nonlinearity of materials.

The different elements of the stiffness matrix can be obtained, according to the procedure exposed by Paz [13], through the expression:

$$k_{ij} = \int_o^L E\cdot I\cdot\psi_i''(x)\cdot\psi_j''(x)\cdot dx \tag{3}$$

Where $\psi_i(x)$, equation (4), is the defining function of the bar deformations $y(x)$ when a unitary displacement δ_I takes place and the rest of displacements are null (see Figure 3).

$$
\begin{aligned}
&\delta_1 = 1 \qquad \psi_1(x) = 1 - 3\cdot\left(\frac{x}{L}\right)^2 + 2\cdot\left(\frac{x}{L}\right)^3\\
&\delta_2 = 1 \qquad \psi_2(x) = x\cdot\left(1-\frac{x}{L}\right)^2\\
&\delta_3 = 1 \qquad \psi_3(x) = 3\cdot\left(\frac{x}{L}\right)^2 - 2\cdot\left(\frac{x}{L}\right)^3\\
&\delta_4 = 1 \qquad \psi_4(x) = \frac{x^2}{L}\cdot\left(\frac{x}{L}-1\right)
\end{aligned} \tag{4}
$$

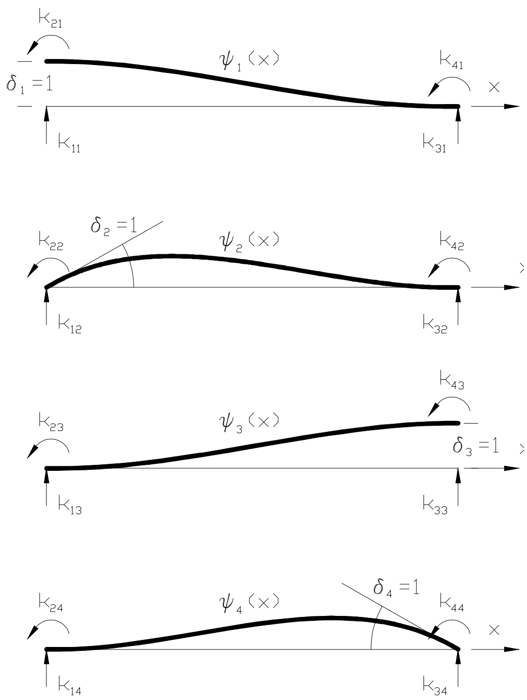

Figure 3: Bar deformations

When the material nonlinearity is wanted to be evaluated, we cannot take the *E·I* term in the equation (3) as a constant value. In this case, the bar stiffness will be obtained from the E·I term evaluation in a set of discrete sections along the bar.

Below is developed an iterative process to obtain the *E·I* modulus of stiffness in reinforced concrete and composites sections while under axial load and biaxial moments.

4 EQUILIBRIUM RESOLUTION OF THE SECTION

4.1 Section response.

In the general case of axial load and biaxial bending, the relation curvature-axial load-moment involves six variables, that in relation to the reference system represented at Figure 4, are: the *P* axial force, Mz and M_Y bending moments, z_n distance from neutral axis to the section centre of gravity, ϕ the plane curvature and the θ angle formed by neutral and Y axes.

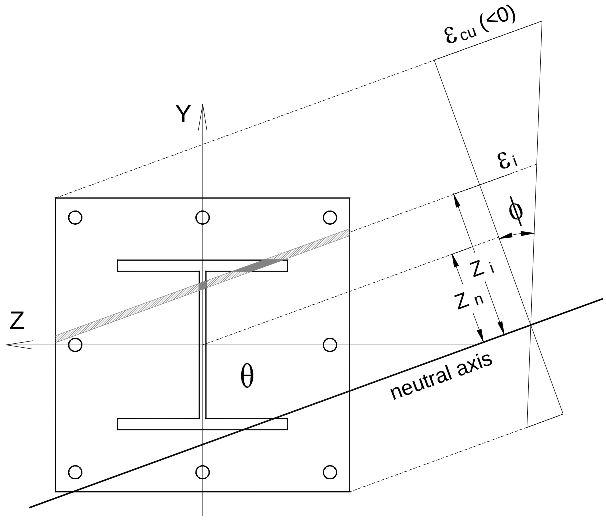

Figure 4 : Section response assessment

Among the most known methods to determine the section response, must be quoted the method that use a section discretization model for segments or cells [8]. Both segments and cells have a constant strength assigned, which is applied to their centre of gravity.

At present work has been used a discretization model of the section based on segments, according to the process described below.

A specific value is assigned to the neutral axis position (θ and z_n) and deformation plane curvature (ϕ), the process begins to change all the points coordinates that defines the section (perimeter vertex, structural section vertex, and the centre of gravity of reinforcing bars), to express them with regard to the reference system defined by a neutral axis. Following coordinates transformation expression is used:

$$\begin{bmatrix} z' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} z \\ y \end{bmatrix} \qquad (5)$$

Where (z’, y’) are the coordinates in the new reference system, θ is the neutral axis rotation angle in relation to Y axis and (z, y) are the point coordinates in relation with Z-Y system.

Then the section is divided in a reduced thickness segments set parallels to the neutral axis. The medium line deformation of each segment is determined by:

$$\varepsilon_i = \phi \cdot y \qquad (6)$$

Each segment tension is obtained by the materials constitutive law application. When a segment involves different materials, it is subdivided into portions for area determination purposes and the stress correspondent to each one of them.

The section response is obtained by numerical integration:

$$\int \sigma(\varepsilon(\phi)) \cdot dA = N$$
$$\int \sigma(\varepsilon(\phi)) \cdot d \cdot dA = M \qquad (7)$$

4.2 Approaching to the solution of the neutral axis position and rotation.

Once we know the axial load and biaxial bending moments, the section equilibrium position determination requires an iterative process.

Although the method developing convergence is guaranteed, the number of iterations required is highly sensitive to the initial values of neutral axis position and rotation. Because of that to get starting values near enough to the final solution is advisable.

When a biaxial moment is acting over a section, which vector coincides with one of his main inertia axis, the neutral axis coincides with the vector before mentioned.

If two biaxial bending moments are acting but the section have the same inertia with respect to its two main axes, then the neutral axis coincides with the vector defined by those two moments. The angle formed with Y axis can obtained through the expression:

$$\theta = \tan^{-1}\left(\frac{M_z}{M_y}\right) \qquad (8)$$

As usually happens, if the section inertias are different in their two axes, then neutral axis will not coincides with the momentum vector, being necessary to correct its position taking into account the inertias.

$$\theta = \tan^{-1}\left(\frac{M_z/I_z}{M_y/I_y}\right) \qquad (9)$$

We will take as superior and inferior limits in neutral axis rotation angle to apply the procedure, will be used:

$$\theta_{\inf} = \theta - \Delta\theta$$
$$\theta_{\sup} = \theta + \Delta\theta \qquad (10)$$

The distance z_n to the section centre of gravity and neutral axis, will depend on the eccentricity of the axial load and area and inertias of the section. Its value can be estimated by the expression:

$$z_n = \frac{1/\Omega}{\sqrt{\left(e_y/I_z\right)^2 + \left(e_z/I_y\right)^2}} \qquad (11)$$

As superior and inferior limits of neutral axis will used:

$$z_{n,\inf} = z_n - \Delta z_n$$
$$z_{n,\sup} = z_n + \Delta z_n \qquad (12)$$

It is rather more complicated to approach the ending curvature of the deformed plane. Because of this, null curvature is adopted as the inferior limit and the curvature correspondent to the section failure surface as the superior limit, for previously defined rotation and position. Medium value is used as an initial value.

$$\phi_{\inf} = 0$$
$$\phi_{\sup} = \phi_{\varepsilon_{cu}}$$
$$\phi = \frac{\phi_{\inf} + \phi_{\sup}}{2} \qquad (13)$$

4.3 Equilibrium position location

The iterative process that is proposed to locate the equilibrium position, based on bisection method, it consists of three loops in chain.

When rotation and position of neutral axis are determined by means of the equations (9) and (11), the first loop locates the curvature of the deformed plane due to the equilibrium between axial load and internal force of the section.

After section response has been obtained, equation (7), with initial curvature, the routine corrects its value at the axial load function. If the section response is bigger than the axial force, then to reduce curvature is necessary, but in the opposite case to increase it. Next iteration value is obtained by the expressions:

$$\begin{aligned} N > N_d &\rightarrow \phi_{\sup} = \phi \\ \phi &= \frac{\phi_{\inf} + \phi_{\sup}}{2} \\ N < N_d &\rightarrow \phi_{\inf} = \phi \\ \phi_m &= \frac{\phi_{\inf} + \phi_{\sup}}{2} \end{aligned} \qquad (14)$$

Second loop determines neutral axis distance to centre of gravity of the section because of the equilibrium of Z axis bending moment with internal reaction of the section. This loop includes the first one, so when convergence is obtained, then neutral axis position will be known, as well as the curvature of deformed plane. After getting section response with initial position of neutral axis (z_n), the routine corrects its value depending on Z axis bending moment. If the section response is bigger than Z axis bending moment, then to move away the neutral axis will be necessary, in the opposite case to move closer. As the section response is obtained in relation to the reference system defined by the neutral axis rotation, to change the reference system to Z-Y axes is necessary. The new value of the neutral axis for the following iteration is obtained by the expressions:

$$\begin{aligned} M_Z > M_{Zd} &\rightarrow z_{n,\inf} = z_n \\ z_n &= \frac{z_{n,\inf} + z_{n,\sup}}{2} \\ M_Z < M_{Zd} &\rightarrow z_{n,\sup} = z_n \\ z_n &= \frac{z_{n,\inf} + z_{n,\sup}}{2} \end{aligned} \qquad (15)$$

Third loop determines neutral axis rotation because of the equilibrium between bending moment in Y axis and the internal moment of the section, keeping a fixed distance from neutral axis to section centre of gravity. This loop includes the two previous loops, so when convergence is obtained, as well the neutral axis rotation and position as the curvature of the deformed plane will be known. After the section response with the initial rotation *(θ)* of neutral axis is obtained, the routine corrects its value in the Y axis bending moment function. If the section response is bigger than Y axis bending moment, then increase of the neutral axis rotation is necessary, in the opposite case, to reduction it. As section response is obtained in relation to the reference system defined by the rotation of the neutral axis, the change from a reference system to Z-Y axes is necessary. The new rotation value of the neutral axis for next iteration is obtained by the expressions:

$$\begin{aligned} M_Y > M_{Yd} &\rightarrow \theta_{\inf} = \theta \\ \theta &= \frac{\theta_{\inf} + \theta_{\sup}}{2} \\ M_Y < M_{Yd} &\rightarrow \theta_{\sup} = \theta \\ \theta &= \frac{\theta_{\inf} + \theta_{\sup}}{2} \end{aligned} \qquad (16)$$

Each loops convergence is achieved when the error between solicitation and the internal section response is considered worthless.

5 CALCULUS ORGANIZATION CHART

Described procedure is below represented by a flow diagram (Figure 5).

6 APPLICATION TO A COMPOSITE SECTION

In order to show the working procedure described before to obtain equilibrium position, a calculation for a 45x45 section has been made with a HEB-260 steel profile encased and 8ϕ12 as reinforcing framework placed with a 3cm layer. The strength of used materials have been f_{ck} = 25 N/mm^2 for concrete, f_{yk} = 275 N/mm^2 for structural steel and f_{sk} = 400 N/mm^2 for reinforcing steel.

As convergence criterion a maximum error coefficient of 2% over the axial load and bending moments has been accepted.

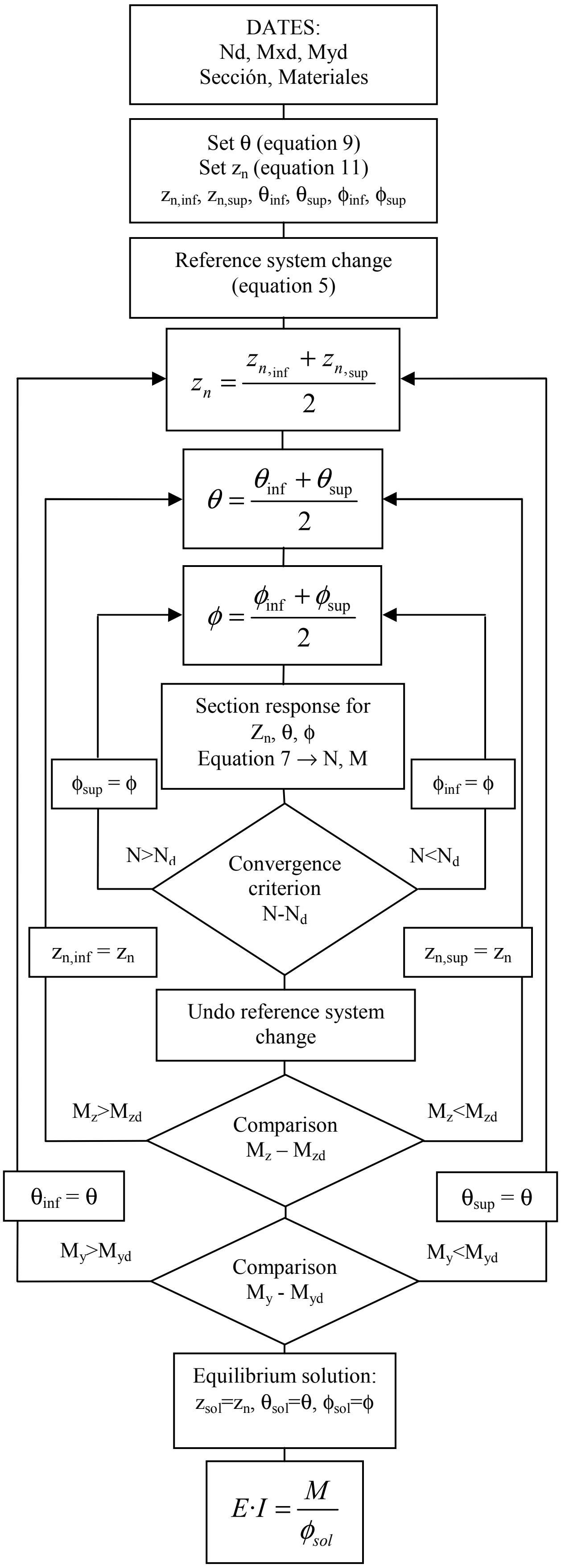

Figure 5: Flow diagram

Table 1 shows the homogenized inertias of the section, together with its axial load and bending moments.

Table 1: Initial values

Ix (10^{-3} m^4)	Iy (10^{-3} m^4)	N_d (kN)	Mx_d (m·kN)	My_d (m·kN)
4,46504	3,77668	1.000	2.000	1.200

Table 2 shows the iterations that have been done to get the convergence between axial load (N_d) and the section response value (described as loop 1).

Table 2: Axial load equilibrium

Iter	ϕ (x10^{-5} rad)	N (kN)	Error %
1	1,4089299	5.890,88	5,89
2	0,704465	3.217,13	3,22
3	0,352232	1.676,49	1,68
4	0,176116	855,22	0,86
5	0,264174	1.270,10	1,27
6	0,220145	1.063,72	1,06
7	0,209138	959,74	0,96
8	0,203634	1.011,80	1,01

The number of iterations is bigger for the first entrance in the loop, due to not to have a curvature value near enough to the solution. For following entrances, where the equilibrium curvature in the previous cycle is already known, the number of iterations is reduced to a medium value of 4.

Table 3 and Table 4 show the iterations that have been done in order to obtain the convergence between the biaxial bending moments (Mz-My) and the response moments of the section when are taking part respectively, in the position (z_n) and in the neutral axis rotation (θ) (described before as 2 and 3 loops).

Table 3: Equilibrium of the moment Mz

Iter	z_n (cm)	Mz (m·kN)	Error %
1	66,49448	3.160,88	1,58
2	76,28089	1.833,97	0,92
3	71,38769	2.040.80	1,02

Table 4: Equilibrium of the moment Y

Iter	θ (degree)	My (m·kN)	Error %
1	54,64969	1.217,94	1,015

It can be observed that initial values, especially rotation (θ) are near enough to the ending solution, due to that the iterations number is small.

7 MOMENT-CURVATURE DIAGRAM

Remaining axial load constant and progressively increasing biaxial bending moments, several curvatures of the deformed plane that lead to the equilibrium of the section can be obtained. The graphical representation of the pair of values M-ϕ, are known as "moment-curvature diagram".

As an example, and using the section described at point 6, has been elaborated the moment-curvature diagram (Figure 6) correspondent to a constant axial load of 1.000 kN and the relation between two bending moments is Mz/My=1,66.

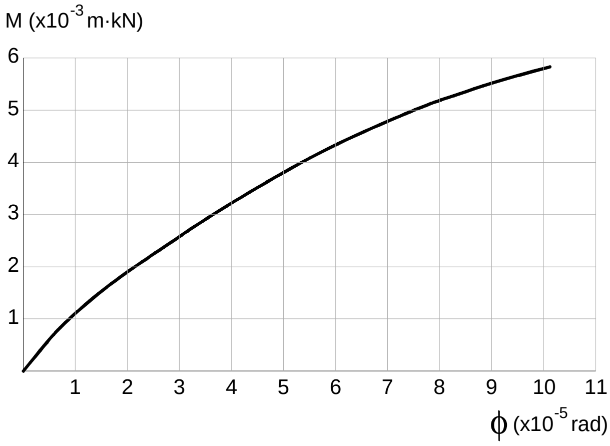

Figure 6 : Moment-curvature diagram (N=1.000kN)

The biggest increasing of the curvature in relation to the moment is due to the nonlinearity between both of them and makes the progressive stiffness loss ($E{\cdot}I$) clear in the section as the bending moment is being increased.

8 CONCLUSIONS

An iterative procedure of quick convergence has been exposed, when starting from the appropriate starting values that allow getting the position of equilibrium of a reinforced concrete or composite section that has been requested for the axial load and biaxial bending moments. In that procedure are taken into account the different strength behaviours of materials that built them up through their nonlinear constitutive laws.

Its implementation on software about nonlinear analysis of structures allows to modify the member stiffness in framed structures taking into account the nonlinear mechanical of the materials, in each loading step.

Although the consideration of the mechanical nonlinearity can be taken into account at any material, it affects significantly to concrete structures due to the variation of its elasticity module depending on the strength applied and the inertia of the section, affected by the material cracking.

Its incidence acquires a special importance in fatigue load of compressed bars design, its measurement is conditioned by the second order effects.

As an example an application of the developed procedure has been shown to obtain the equilibrium of a steel concrete encased section. The tables show the small number of iterations needed to obtain the section equilibrium. This together with the current powerful computers allows quickly enough the approaching to mechanical nonlinear design of concrete structures.

Finally, the moment-curvature diagram of the section for a constant axial load of 1.000 kN has been represented.

The shown procedure represents a powerful and quick method to structural design taking into consideration the mechanic nonlinear of materials and an effective tool to the future researches developing.

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